Evil twins alternate with odious twins

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An integer is called *evil* if the number of ones in its binary expansion is even and *odious* if the number of ones in the binary expansion is odd. If we look at the integers between 0 and 15 we find that $0, 3, 5, 6, 9, 10, 12, 15$ are evil and that $1, 2, 4, 7, 8, 11, 13, 14$ are odious.

Next we say that if two consecutive integers are evil then this is a pair of *evil twins* and that if two consecutive integers are odious then this is a pair of *odious twins*. Returning to the integers from 0 to 15 we see that $\{5, 6\}$ and $\{9, 10\}$ are two sets of evil twins and that $\{1, 2\}$, $\{7, 8\}$ and $\{13, 14\}$ are three pairs of odious twins. We can now state the new result of this paper:

Theorem (Evil Twin)**.** *Evil twins alternate with odious twins.*

The terminology of evil and odious is fairly new coming from combinatorial game theory [2] , but the theory connected to these numbers has many applications and a long history. One of the first results in the area is due to Prouhet [8].

If we look at the numbers from 0 to 15 we see that the number of evil numbers equals the number of odious numbers and that the sum of the evil numbers equals the sum of the odious numbers. More surprisingly the sum of the squares of the evil numbers equals the sum of the squares of the odious numbers and the sum of the cubes of the evil numbers equals the sum of the cubes of the odious numbers. If we let $0^0 = 1$ we can write the preceding statements as

$$1^0+2^0+4^0+7^0+8^0+11^0+13^0+14^0 = 0^0+3^0+5^0+6^0+9^0+10^0+12^0+15^0$$

$$1^1+2^1+4^1+7^1+8^1+11^1+13^1+14^1 = 0^1+3^1+5^1+6^1+9^1+10^1+12^1+15^1$$

$$1^2+2^2+4^2+7^2+8^2+11^2+13^2+14^2 = 0^2+3^2+5^2+6^2+9^2+10^2+12^2+15^2$$

$$1^3+2^3+4^3+7^3+8^3+11^3+13^3+14^3 = 0^3+3^3+5^3+6^3+9^3+10^3+12^3+15^3$$

More succinctly we can write the preceding statements as

$$\sum_{\substack{k=0 \\ k \text{ is evil}}}^{15} k^j = \sum_{\substack{k=0 \\ k \text{ is odious}}}^{15} k^j \text{ for } 0 \leq j \leq 3.$$

Prouhet proved the remarkable result:

Theorem (Prouhet)**.**

$$\sum_{\substack{k=0\\ k \text{ is evil}}}^{2^n-1} k^j = \sum_{\substack{k=0\\ k \text{ is odious}}}^{2^n-1} k^j \text{ for } 0 \le j \le n-1.$$

In fact Prouhet proved a more general theorem. He looked at integers written in any given base b, and then partitioned $0, 1, \ldots b^n - 1$ according to the sum of their digits modulo b. He showed that the sums of the integers raised to the power j in each class was equal for any j satisfying $0 \le j \le n-1$. There have been generalizations of this theorem. Lehmer [5] proved a generalization in 1947 and then Sinha [10] generalized it further in 1972.

We will give the proof of the version Prouet's Theorem that we have stated above later, but a good place to start our study is with the Thue-Morse sequence.

The Thue-Morse sequence

For an integer n will define $\alpha(n) = 1$ if n is evil and $\alpha(n) = -1$ if n is odious. Then the *Thue-Morse sequence* is $\alpha(0), \alpha(1), \alpha(2), \ldots$. So the Thue-Morse sequence begins

$$1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, \ldots$$

Often we will slightly simplify the notation by omitting the ones and just writing the signs; so the sequence will be written as

$$+ - - + - + + - - + + - + - - + \cdots$$

Notice that the sequence has a nice property that after the first term $+$ comes the negation of that term $-$, that after the first two terms $+-$ comes the negation of those terms $-+$, after the first four terms $+--+$ comes the negation $-++-$. In general the terms from 2^n to $2^{n+1}-1$ is just the negation of the terms from 0 to $2^n - 1$. This follows from the fact that if $0 \le k \le 2^n - 1$ then the binary expansion of $2^n + k$ has one more 1 in its binary expansion than does k (namely it has an extra 1 in the 2^n place) and so $\alpha(k)$ and $\alpha(2^n + k)$ have opposite signs. This property makes it very easy to write down the sequence, and will be important later when we come to generating functions.

The Thue-Morse sequence has several other useful properties and has been used in a variety of areas of mathematics. We will briefly discuss some of these. For a more complete list of applications and history the reader should see [1].

One important area where the series occurs is in the application of symbolic dynamics to dynamical systems. Consider the map

$$f(x) = 4x(1-x).$$

We look at points $x \in [0,1]$ and see what we can say about the sequence

$$x, f(x), f^2(x), f^3(x), \ldots$$

where f^i denotes f composed (not multiplied) with itself i times. Now define

$$\beta(x) = \begin{cases} -1 & \text{if } 0 \le x < .5 \\ c & \text{if } x = .5 \\ 1 & \text{if } .5 < x \le 1. \end{cases}$$

i.e., β tells us whether x is to the left or right of .5 or equal to .5. To greatly simplify things instead of considering our sequence

$$x, f(x), f^2(x), f^3(x), \ldots$$

we consider the string of -1s and 1s and possibly a c given by

$$\beta(x), \beta(f(x)), \beta(f^2(x)), \beta(f^3(x)), \ldots.$$

It turns out that for any sequence of -1s and 1s we can find an x with exactly that sequence. The Thue-Morse sequence then shows that there must be a point x that is not periodic and not eventually periodic. (A good introduction to symbolic dynamics is [3] Chapter 1.6.) This lack of periodicity was the property that Morse used the sequence for in proving a result in differential geometry about geodesics being recurrent but non-periodic on certain surfaces of negative curvature [6].

The Thue-Morse sequence has the property that it contains examples of blocks of symbols X such that XX occurs in the sequence. For example, if we take X to be $+$ we find in the sequence $++$ as consecutive terms, if we take X to be $+-$ we can find $+-+-$ as consecutive terms in the sequence. Of course we have to be careful how we choose X. If we take X to be $++$ we cannot find $++++$ as consecutive terms in the sequence, but the important point is that there do exist some blocks X such that XX occurs in the sequence. More interestingly, Axel Thue [11] and [12] (both reprinted in [7]) showed that there is no block of symbols X such that XXX occurs. This is sometimes referred to as saying that the Thue-Morse sequence is cube-free. Max Euwe, the Dutch chess grandmaster and mathematician, used this property to show that there could be infinite games of chess in which the same sequence of moves never occurred three times in succession [4]. (In particular, for us, this cube-free property means that there are no evil or odious triplets.)

One important way of proving results about this sequence is by looking at its generating function. This is what we do next.

The Thue-Morse generating function

Notice that $(1-x)(1-x^2) = 1 - x - x^2 + x^3$ and that the coefficients of the polynomial on the right are just the first four terms of the Thue-Morse sequence. If we take $1 - x - x^2 + x^3$ and multiply by $1 - x^4$ we obtain a polynomial whose first four coefficients remain as before, and the coefficients of x^4 to x^7 are just the first four coefficients with the signs reversed. So we end up with a polynomial of degree 7 whose coefficients are the first eight terms in the Thue-Morse sequence. Inductively we can show that

$$\prod_{k=0}^{n-1}(1 - x^{2^k}) = \sum_{k=0}^{2^n-1} \alpha(k)x^k$$

and thus obtain our generating function

$$\prod_{k=0}^{\infty}(1 - x^{2^k}) = \sum_{k=0}^{\infty} \alpha(k)x^k. \tag{1}$$

We now begin our study of twins. Suppose we multiply both sides of the equation above by $1 + x$. We obtain

$$\begin{aligned}(1+x)\prod_{k=0}^{\infty}(1 - x^{2^k}) &= (1+x)\sum_{k=0}^{\infty} \alpha(k)x^k \\ &= \sum_{k=0}^{\infty} \alpha(k)(x^k + x^{k+1}) \\ &= 1 + \sum_{k=1}^{\infty}(\alpha(k) + \alpha(k-1))x^k.\end{aligned}$$

Notice that $\alpha(k)+\alpha(k-1)$ will take values of $+2, 0$ or -2 depending on whether $k-1$ and k are evil twins, not twins or odious twins, respectively.

For small values of n it is easy to compute $(1+x)\prod_{k=0}^{n-1}(1-x^{2^k})$ using a computer algebra system. It is suggested that the reader try various examples. For example,

$$(1+x)\prod_{k=0}^{3}(1 - x^{2^k}) = 1 - 2x^2 + 2x^6 - 2x^8 + 2x^{10} - 2x^{14} + x^{16}$$

which tells, us as we observed before, that $\{5, 6\}$ and $\{9, 10\}$ are sets of evil twins and that $\{1, 2\}$, $\{7, 8\}$ and $\{13, 14\}$ are three pairs of odious twins.

In the next two short sections we will give proofs of both the theorems stated in the introduction. Both follow from the generating function of the Thue-Morse sequence.

Proof that evil and odious twins alternate

We consider

$$(1+x)\prod_{k=0}^{\infty}(1-x^{2^k}) = 1 + \sum_{k=1}^{\infty}(\alpha(k)+\alpha(k-1))x^k.$$

As was stated before, with the exception of the first term, we know that the coefficients are $+2, 0$ or -2 depending on whether $k-1$ and k are evil twins, not twins or odious twins, respectively. Now

$$\begin{aligned}(1+x)\prod_{k=0}^{\infty}(1-x^{2^k}) &= (1-x^2)\prod_{k=1}^{\infty}(1-x^{2^k}) \\ &= (1-x^2)\prod_{k=0}^{\infty}(1-(x^2)^{2^k}) \\ &= (1-x^2)\sum_{k=0}^{\infty}\alpha(k)(x^2)^k \\ &= \sum_{k=0}^{\infty}\alpha(k)((x^2)^k-(x^2)^{k+1}) \\ &= 1+\sum_{k=1}^{\infty}(\alpha(k)-\alpha(k-1))x^{2k}. \qquad (2)\end{aligned}$$

We see that the coefficient of x^k is zero if k is odd. So that for $\{k-1, k\}$ to be a pair of twins k must be even. (We didn't really need such a complicated argument to deduce this as it is clear that if $k-1$ is even then its last binary digit must be 0 so $\alpha(k-1)$ and $\alpha(k)$ must have opposite signs.) The more interesting conclusion is that $\{2k-1, 2k\}$ is an evil pair if and only if $\alpha(k) = 1$ and $\alpha(k-1) = -1$; and $\{2k-1, 2k\}$ is an odious pair if and only $\alpha(k) = -1$ and $\alpha(k-1) = +1$. This means that as we look along the Thue-Morse sequence whenever we see $-+$ in the $k-1$, k positions we know that $2k-1$ and $2k$ will be evil twins; and whenever we see $+-$ in the $k-1$, k positions we know that $2k-1$ and $2k$ will be odious twins; and there are no other sets of twins. Clearly, any finite string that begins and ends with $+-$ must contain a

$-+$, and any finite string that begins and ends with $-+$ must contain a $+-$. So evil twins must alternate with odious twins.

Prouhet's Theorem

In this final section we give a proof of Prouhet's theorem. There have been many proofs of this. We follow Roberts [9] and Wright [13] using difference operators. ([13] contains interesting history about this theorem and the connection with the Tarry-Escott problem.)

Given any polynomial in one variable, $P(x)$ say. Let E denote the operator defined by $E(P(x)) = P(x+1)$. So E can be thought of as translating the graph of $P(x)$ one unit horizontally. For any positive iteger m we let E^m denote E composed with m times. Thus $E^m(P(x)) = P(x+m)$. We also let I denote the identity operator, $I(P(x)) = P(x)$.

The key observation is that if $P(x)$ is a polynomial of degree d then

$$(I - E^m)(P(x)) = I(P(x)) - E^m(P(x)) = P(x) - P(x+m)$$

is a polynomial of degree $d-1$, and that if $P(x)$ is a constant, $P(x) = c$, then

$$(I - E^m)(P(x)) = I(P(x)) - E^m(P(x)) = c - c = 0.$$

We can now re-write the Thue-Morse generating function using the operator E. We obtain

$$\prod_{k=0}^{n-1}(I - E^{2^k}) = \sum_{k=0}^{2^n-1} \alpha(k)E^k.$$

Suppose that $P(x)$ is a polynomial, then

$$\prod_{k=0}^{n-1}(I - E^{2^k})P(x) = \sum_{k=0}^{2^n-1} \alpha(k)E^k P(x).$$

Suppose that $P(x)$ has degree $d \leq n-1$, then by the key observation above we see that $\prod_{k=0}^{n-1}(I - E^{2^k})P(x) = 0$ and so we obtain

$$0 = \sum_{k=0}^{2^n-1} \alpha(k)E^k P(x) = \sum_{k=0}^{2^n-1} \alpha(k)P(x+k).$$

Letting $P(x) = x^j$, for $0 \leq j \leq n-1$ gives

$$0 = \sum_{k=0}^{2^n-1} \alpha(k)(x+k)^j.$$

Finally, putting $x=0$ and re-arranging gives

$$\sum_{\substack{k=0 \\ k \text{ is evil}}}^{2^n-1} k^j = \sum_{\substack{k=0 \\ k \text{ is odious}}}^{2^n-1} k^j \text{ for } 0 \leq j \leq n-1,$$

which completes the proof.

Acknowledgment. We thank the anonymous referees for their many helpful suggestions.

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