ROTATION MATRICES FOR VERTEX MAPS ON GRAPHS

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Abstract. Let $G$ be a finite connected graph. Suppose $f : G \rightarrow G$ is a map homotopic to the identity that permutes the vertices. For such a map, a rotation matrix is defined and the basic properties of this matrix are given. It is shown that this matrix generalizes some of the information given by the rotation interval, which is defined when the graph is a circle, to more general graphs.

1. Introduction

In one-dimensional combinatorial dynamics, the two basic starting points are looking at maps on the interval and maps on the circle. For maps of the interval, the basic result is Sharkovsky’s Theorem. For maps of the circle, the basic results concern the rotation interval for maps that have degree one. In [3] and [4], Sharkovsky-type theorems were given for maps on trees and graphs. Generalizing the rotation interval to maps on graphs has proved difficult. This paper introduces a matrix that gives some of the information that rotation intervals give for degree-one circle maps for the special case when we have a map on the graph that permutes the vertices.

Given a vertex map on the graph, the matrix can be calculated simply from its oriented Markov matrix. The definition of the oriented Markov matrix and some of the properties of powers this matrix can be found in [4], but for clarity and ease of exposition the basic facts are given again in Sections 2 to 6 Following these, in Section 7, an simple example of degree-one map of a circle is given that illustrates the information that the rotation interval yields. The rotation matrix is then defined, some basic properties are proved, and it is shown that it gives the same type of information as the rotation interval.

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2. Graphs

An edge is a space homeomorphic to the closed interval \([0, 1]\). The boundary points of the edges are vertices. An edge is not allowed to have a vertex as an interior point. The intersection of two distinct edges is empty, consists of one vertex or of two vertices. We assume that we have a finite number \(e\) of edges and \(v\) of vertices. The graph is the union of vertices and edges. We assume that graphs are connected. We allow the possibility that there is more than one edge between the same two vertices. We also allow the possibility that an edge connects a vertex to itself (in this case the edge is homeomorphic to a circle).

We will label the vertices \(V_1, \ldots, V_v\); in diagrams we will just label them with the integers 1 to \(v\). If \(V_k\) and \(V_l\) are vertices and there is an edge between them we will choose an orientation or direction on the edge and consider it as a directed edge from one vertex to the other. We will label the positively oriented edges as \(E_1, \ldots, E_e\). If \(E_p\) is an edge from \(V_k\) to \(V_l\) then the same edge, but with the opposite (negative) orientation that goes from \(V_l\) to \(V_k\) will be denoted by \(-E_p\). We will call \(V_k\) the first vertex of \(E_p\) and \(V_l\) the second vertex. (So the first vertex of \(E_p\) equals the second vertex of \(-E_p\) and vice versa.)

Given any two vertices, \(V_r\) and \(V_s\), a path from \(V_r\) to \(V_s\) is a sequence of edges \(E_1 \ldots E_q\) where the first vertex of \(E_1\) is \(V_r\), the second vertex of \(E_q\) is \(V_s\) and the second vertex of \(E_l\) equals the first vertex of \(E_{l+1}\) for \(1 \leq l \leq q - 1\). If \(E_p\) and \(-E_p\) are two consecutive edges in a path we can obtain a shorter path by omitting these two edges. We will call this a contraction of the path. Given any path from vertex \(V_r\) to vertex \(V_s\) we can form a sequence of contractions resulting in a unique path that cannot be contracted further. We call this resulting path fully contracted. Given a path \(P\) in the graph we will let \(\text{fc}(P)\) denote the fully contracted path that is obtained from \(P\).

We define a circle in graph \(G\) to be a closed path that has no repeated vertices. Thus in Figure 2, \(E_1 E_2 - E_3\) and \(E_3 E_4 E_5 E_6\) are circles.

3. Linearization of vertex maps

A vertex map of a graph \(G\) is a continuous map from \(G\) to itself with the property that the vertices of \(G\) are permuted by \(f\). In this section we will define the linearization of a vertex map.

Suppose \(\theta\) is a permutation on \(1, \ldots, v\) and that \(f\) is a map from graph \(G\) to itself with the property that \(f(V_i) = V_{\theta(i)}\) for \(1 \leq i \leq v\). Then we define \(L_f\), the linearization of \(f\), to be the continuous map from \(G\) to itself given by \(L_f\) maps the edge \(E\) with first vertex \(V_k\) and second vertex \(V_l\) linearly onto the fully contracted path from \(V_{\theta(k)}\) to
Figure 1. Circle map with vertex permutation $(1\,3)(2\,4\,5)$

For maps of the interval or of trees this linearized map is often called the connect-the-dot map associated to $\theta$. For trees (and intervals) $f$ is always homotopic to the identity and so there is a unique linearization associated to $\theta$. However, for graphs in general, the linearization depends on both $\theta$ and the homotopy type of the map $f$.

The linearization of $f$ gives a well-defined function from paths in graph $G$ to paths in $G$. We illustrate this in the following example.

We give two examples that we will refer to throughout the paper.

Example 1. Figure 1 shows a circle $G$ with 5 vertices and 5 edges. The map $f : G \to G$ permutes the vertices by $f(V_i) = V_{\theta(i)}$ where $\theta \in S_5$ is given by $\theta = (1\,3)(2\,4\,5)$, and $f(E_1) = E_3$, $f(E_2) = E_4E_5$, $f(E_3) = -E_5$, $f(E_4) = E_5E_1$, $f(E_5) = E_2$.

In this case we have: $f^2(E_1) = -E_5$, $f^3(E_1) = -E_2$, $f^4(E_1) = -E_5 - E_4$, and $f^5(E_1) = -E_2 - E_1 - E_5$. 

Figure 2. Graph with vertex permutation $(1\,3)(2\,4\,5)$
Example 2. Figure 2 shows a graph $G$. In this case the map $f : G \rightarrow G$ permutes the vertices by $f(V_i) = V_{\theta(i)}$ where $\theta \in S_n$ is given by $\theta = (1 3)(2 4 5)$, and $f(E_1) = -E_3E_1E_2E_4$, $f(E_2) = E_5$, $f(E_3) = E_4E_5$, $f(E_4) = E_6E_1$, $f(E_5) = -E_1$ and $f(E_6) = E_3$.

It is straightforward to check that

$f^3(E_1) = E_1 - E_1 - E_6 - E_5 - E_4 - E_3E_1E_2E_4E_5E_6E_1 - E_1E_3 - E_3E_1E_2E_4$

and

$L_{f^3}(E_1) = -E_6 - E_5 - E_4 - E_3E_1E_2E_4E_5E_6E_1E_2E_4$.

4. Oriented Markov Graphs

We now define the Oriented Markov Graph associated to the graph $G$ and $L_f$. This is a directed graph on $e$ vertices. Each of the directed edges also has either a positive or negative sign attached. It is defined as follows: the vertices of the Markov graph correspond to the edges of the graph (we will abuse notation and also denote these vertices by $E_1, \ldots, E_e$); we draw a positive directed edge from vertex $E_i$ to vertex $E_j$ for each occurrence of $E_j$ in $L_f(E_i)$; we draw a negative directed edge from vertex $E_i$ to vertex $E_j$ for each occurrence of $-E_j$ in $L_f(E_i)$. We will denote this Oriented Markov Graph by $OMG(f)$.

Sometimes we will not need to use the fact that the directed edges in $OMG(f)$ have a positive or negative sign attached. We will use the notation $E_k \rightarrow E_j$ to indicate that there is a directed edge from $E_k$ to $E_j$.

Let $E_{k_1} \rightarrow E_{k_2} \rightarrow \ldots \rightarrow E_{k_m}$ be a path in the Oriented Markov Graph. This path will be have positive orientation if the number of negative directed edges is even and negative orientation if the number of negative directed edges is odd.

Let $S_v$ denote the symmetric group on $v$ letters. The following is immediate:

Lemma 1. Let $\theta \in S_v$. Let $G$ be a graph with $v$ vertices $V_1, V_2, \ldots, V_v$. Let $f$ be a map from graph $G$ to itself with the property that $f(V_i) = V_{\theta(i)}$ for $1 \leq i \leq v$. Suppose that $E_{k_1} \rightarrow E_{k_2} \rightarrow \ldots E_{k_m}$ is a path in $OMG(f)$. Then $E_{k_m} \subseteq L_f^m(E_{k_1})$. If the path $E_{k_1} \rightarrow E_{k_2} \rightarrow \ldots E_{k_m}$ in $OMG(f)$ has positive orientation then $E_{k_m}$ appears in the path (in $G$) $L_f^m(E_{k_1})$. If the path $E_{k_1} \rightarrow E_{k_2} \rightarrow \ldots E_{k_m}$ in $OMG(f)$ has negative orientation then $-E_{k_m}$ appears in the path (in $G$) $L_f^m(E_{k_1})$.

Conversely, if $E_{k_m}$ appears in the path (in $G$) $L_f^m(E_{k_1})$ then there is a positively oriented path (in $OMG(f)$) of length $m$ from $E_{k_1}$ to
Let \( E_{k_0} \) be a vertex of \( G \) (to be the property that \( f(\theta) = V_{\theta(i)} \) for \( 1 \leq i \leq v \). Suppose that \( E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0} \) is a loop in \( OMG(f) \). Then there exists a periodic point \( x \) of \( L_f \) with \( L_f^{m+1}(x) = x \) such that \( L_f^r(x) \in E_{k_r} \) for \( r = 0, 1, \ldots, m \). Conversely, if \( x \) is a periodic point in \( L_f \) of period \( m + 1 \) and if \( x \) is not a vertex then there exists a loop \( E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0} \) in \( G(\theta) \) such that \( L_0^r(x) \in E_{k_r} \) for \( r = 0, 1, \ldots, m \).

As in [3] we can change \( L_f \) to \( f \) in the first half of the above lemma to obtain the following.

Lemma 3. Let \( \theta \in S_v \). Let \( G \) be a graph with \( v \) vertices \( V_1, V_2, \ldots, V_v \). Let \( f \) be a map from graph \( G \) to itself with the property that \( f(\theta) = V_{\theta(i)} \) for \( 1 \leq i \leq v \). Suppose that \( E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0} \) is a loop in \( OMG(f) \). Then there exists a periodic point \( x \) of \( f \) with \( f^{m+1}(x) = x \) such that \( f^r(x) \in E_{k_r} \) for \( r = 0, 1, \ldots, m \).

If the loop \( E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0} \) in the above statement is not a repetition of a shorter loop then we will say that it is a non-repetitive loop of length \( m + 1 \).

The importance of non-repetitive loops in the Oriented Markov Graph is the following extension of Lemma 2.

Lemma 4. Let \( \theta \in S_v \). Let \( G \) be a graph with \( v \) vertices \( V_1, V_2, \ldots, V_v \). Let \( f \) be a map from graph \( G \) to itself with the property that \( f(\theta) = V_{\theta(i)} \) for \( 1 \leq i \leq v \). Suppose that \( E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0} \) is a non-repetitive loop in \( OMG(f) \). Then there exists a periodic point \( x \) of \( f \) with \( f^{m+1}(x) = x \) such that \( f^r(x) \in E_{k_r} \) for \( r = 0, 1, \ldots, m \). If \( x \) is not a vertex of \( G \), then \( x \) has minimum period of \( m + 1 \).

5. Oriented Markov Matrices

Suppose \( \theta \in S_v \) and that \( f \) is a map from graph \( G \) to itself with the property that \( f(\theta) = V_{\theta(i)} \) for \( 1 \leq i \leq v \). Then we define the Oriented Markov Matrix of \( f \), denoted \( M(f) \), to be the \( e \times e \) matrix with \( M(f)_{i,j} \) equal to the number of positive directed edges from \( E_j \) to \( E_i \) minus the number of the negative edges from \( E_j \) to \( E_i \). This is equivalent to saying that \( M(f)_{i,j} \) is equal to the sum of the number of
times that $E_i$ appears in the path $L_f(E_j)$ minus the number of times that $-E_i$ appears in the path $L_f(E_j)$.

**Example 1 continued.** In Example 1 the oriented Markov matrix is

$$M(f) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}.$$  

**Example 2 continued.** In Example 2 the oriented Markov matrix is

$$M(f) = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$  

Induction gives the following, where the notation $M^n(f)$ is used to denote $(M(f))^n$, and $M^n(f)_{i,j}$ denotes the $ij$ entry in $(M(f))^n$.

**Theorem 1.** Suppose $\theta \in S_v$ and that $f$ is a map from graph $G$ to itself with the property that $f(V_i) = V_{\theta(i)}$ for $1 \leq i \leq v$. Then $M^n(f)_{i,j}$ is the number of positively oriented paths of length $n$ (in $OMG(f)$) from $E_j$ to $E_i$ minus the sum of negatively oriented paths of length $n$ (in $OMG(f)$) from $E_j$ to $E_i$.

Observe that to obtain $L_{L_f^n}$ from $L_f^n$ we contract paths $L_f^n(E_i)$ to $FC(L_f^n(E_i))$ and that each time a contraction is formed we lose the same edge with both positive and negative orientation, so for each $j$ the number of times $E_j$ appears minus the number of times that $-E_j$ appears is the same in both $L_f^n(E_i)$ and $FC(L_f^n(E_i))$. So $M(L_{L_f^n}) = M(L_f^n)$.

**Theorem 2.** Suppose $\theta \in S_v$ and that $f$ is a map from graph $G$ to itself with the property that $f(V_i) = V_{\theta(i)}$ for $1 \leq i \leq v$. Then $M^n(f) = M(L_f^n) = M(L_{L_f^n})$ for any positive integer $n$.

### 6. Maps homotopic to the identity

Given a vertex map that is homotopic to the identity we will let $h$ denote a specific homotopy. So $h : G \times I \to G$, where $I$ is the unit interval, with $h(x, 0) = x$ and $h(x, 1) = f(x)$ for all $x \in G$. We will use the homotopy to give us paths from each vertex $V_i$ to $f(V_i)$. 


Thus in Example 1 a choice for $h$ could be given by $h(V_1, t) = E_1 E_2$, $h(V_2, t) = E_2 E_3$, $h(V_3, t) = E_3 E_4 E_5$, $h(V_4, t) = E_4$ and $h(V_5, t) = E_5 E_1$.

And for Example 2 a choice for $h$ is given by $h(V_1, t) = E_2 E_4$, $h(V_2, t) = E_4 E_5$, $h(V_3, t) = E_5 E_6 E_1$, $h(V_4, t) = E_6$ and $h(V_5, t) = E_3$.

To simplify notation we will denote the fully contracted path from $V_i$ to $f^k(V_i)$ given by the homotopy by $P^k(V_i)$. The following lemma will be important in what follows. The proof is immediate.

Lemma 5. Let $f : G \to G$ be a map homotopic to the identity that permutes the vertices. Let $E_s$ be an edge in $G$ that goes from vertex $V_i$ to $V_j$. Let $h$ be a homotopy. Then $L_f(E_s) = f c((−P(V_i))E_s P(V_j))$ and $L_{f^k}(E_s) = f c(L_f^k(E_s)) = f c((−P^k(V_i))E_s P^k(V_j))$.

7. Rotation intervals for maps of the circle

In this section we will refer back to the map in Example 1. Note that this map is of degree one (i.e. homotopic to the identity). The usual definition of the rotation number of a point starts by first choosing a lift of the map to the universal cover. Different liftings can give different values. A different lifting, or equivalently, choosing a different homotopy, can shift the rotation interval by an integer, see [1]. For the example under consideration, a lifting can be chosen so that the rotation number associated to 1 is $1/2$ and the rotation number associated to 2 is $1/3$. The rotation interval associated to $L_f$ is $[1/3, 1/2]$. A basic result about rotation intervals is (see [1]):

Theorem 3. Let $f$ be a degree one map of a circle with rotation interval $[a, b]$. Then for any $c$ satisfying $a < c < b$ there exists a point in the circle with rotation number equal to $c$.

In the example under consideration, we know that for any rational number $c$, $1/3 < c < 1/2$, there exists a periodic point with rotation number $c$. Notice that if $a$ and $b$ are rational numbers, then any rational $c$ satisfying $a < c < b$ satisfies $c = \frac{1}{r+s}(ra+sb)$ for some positive integers $r$ and $s$, and for any positive integers $r$ and $s$ we have $a < \frac{1}{r+s}(ra+sb) < b$.

The rotation number of 1 being 1/2 means that under every two iterations of the map the point 1 travels once around the circle (this depends on the particular lift – or equivalently – homotopy chosen). Similarly, the point 2 travels once around the circle under every three iterations. Consequently, after every six iterations 1 has traveled 3 times around the circle and 2 has traveled 2 times around the circle. So under six iterations $E_1$ gets sent to $-E_2-3-E_4-E_5$ and $E_2$ gets sent to $E_2E_3E_4E_5E_1E_2$, but these two statements are independent of
the particular lifting (particular homotopy) and can be determined by looking at \( M^6(L_f) \).

**Example 1 continued.** In Example 1 the oriented Markov matrix is

\[
M^6(f) = \begin{pmatrix}
0 & 1 & -1 & 0 & 1 \\
-1 & 2 & -1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 1 \\
-1 & 1 & -1 & 0 & 1 \\
-1 & 1 & -1 & 0 & 2
\end{pmatrix}.
\]

Let \( s \geq 2 \) and let \( f \) be a vertex map on a graph \( G \). An \( s \)-horseshoe for \( f \) is defined to be an interval \( J \) such that \( J \) is contained in one of the edges with \( s \) subintervals \( J_1, J_2, \ldots, J_s \subset J \) with disjoint interiors such that \( f(J_i) = J \) for \( 1 \leq i \leq s \).

If there is an \( s \)-horseshoe then the topological entropy of \( L_f \) is greater than or equal to \( \log s \), see [1] Proposition 4.3.2.

In the example above we can see that \( E_2 \) (and \( E_5 \)) covers itself twice under \( L_6^6 \). This means that \( G \) contains a 2-horseshoe under \( L_6^6 \). Thus the topological entropy of \( L_6^6 \) is greater than or equal to \( \log 2 \), and so the topological entropy of \( L_f \) (and also \( f \)) is greater than or equal to \( \frac{1}{6} \log 2 \).

As noted above, for Example 1 the rotation interval is \([1/3, 1/2]\). Rational points in this interior of this interval correspond to positive integers \( r \) and \( s \). We also know that we have a 2-horseshoe under \( L_6^6 \). We can choose a periodic point of period \( r + s \) under \( L_6^6 \) that has \( r \) iterates in the first lap and \( s \) iterates in the second lap. It is a straightforward calculation to show that the rotation number of this point is \( \frac{1}{r+s}(r/3 + s/2) \).

It is seen that for degree 1 maps of the circle that the existence of rotation interval (that is not just a single number) is equivalent to having an interval within the circle that has an image under an iterate of the map that contains the whole circle, and that this implies the existence of horseshoes under some iterate of the map. The next section defines a matrix that gives a natural way of detecting this information.

### 8. The rotation matrix

Let \( f \) be a vertex map on a graph \( G \) that is homotopic to the identity with vertex permutation \( \theta \in S_v \). The rotation matrix associated to \( L_f \) is \( R(L_f) = \frac{1}{n}(M^n(L_f) - I) \), where \( I \) is the identity matrix and \( n \) is any positive integer such that \( \theta^n \) is the identity.

The following theorem shows that \( R(L_f) \) is well-defined and does not depend on the particular value of \( n \).
**Theorem 4.** Let $f$ be a vertex map on a graph $G$ that is homotopic to the identity with vertex permutation $\theta \in S_n$. Let $m$ be the smallest positive integer such that $\theta^m$ is the identity. Let $A = \frac{1}{m}(M^m(L_f) - I)$, then $M^{km}(L_f) = I + kmA$ for any $k \in \mathbb{Z}$.

**Proof.** By Lemma 5, if $E_s$ is an edge in $G$ that goes from vertex $V_i$ to $V_j$, then $\text{fc}((L_f^m)^m(E_s)) = \text{fc}((P^m(V_i))E_sP^m(V_j))$. Since $\theta^m$ is the identity, both $P^m(V_i)$ and $P^m(V_j)$ must be closed loops in $G$.

By definition $M^m(L_f) = I + mA$. Since $M^m(L_f)_{t,s}$ is the number of times $E_t$ appears in $L_f^m(E_s)$ minus the number of times that $-E_t$ appears in $L_f^m(E_s)$, it is clear that $(mA)_{t,s}$ is the number of times $E_t$ appears in $(-P^m(V_i))P^m(V_j)$ minus the number of times that $-E_t$ appears in $(-P^m(V_i))P^m(V_j)$.

Let $k$ be a positive integer. Then

$$I^{km}_f(E_i) = \text{fc}((-P^km(V_i))E_sP^km(V_j)).$$

As before $(M^{km}(L_f) - I)_{t,s}$ equals the number of times $E_t$ appears in $(-P^km(V_i))P^km(V_j)$ minus the number of times that $-E_t$ appears in $(-P^km(V_i))P^km(V_j)$. But both $P^m(V_i)$ and $P^m(V_j)$ are loops and so $P^km(V_i)$ and $P^km(V_j)$ consist of $k$ repetitions of $P^m(V_i)$ and $P^m(V_j)$, respectively. Thus $(M^{km}(L_f) - I)_{t,s} = k(mA)_{t,s} = (kmA)_{t,s}$. This completes the proof for positive $k$.

Notice that $M^m(L_f) = I + mA$ and so squaring both sides gives $M^{2m}(L_f) = (I + mA)^2 = I + 2mA + m^2A^2$. But we have shown that $M^{2m}(L_f) = I + 2mA$. So $A^2$ must be the zero matrix. Thus for any positive $k$ we have $M^{km}(L_f)(I - kmA) = (I + kmA)(I - kmA) = I$. This implies that $M^{-km}(L_f)$ exists and equals $I - kmA$. This completes the proof for negative $k$. The proof for $k = 0$ is trivial. \hfill \Box

In the proof above we have that $A^2 = 0$. An immediate consequence is the following corollary.

**Corollary 5.** Let $f$ be a vertex map on a graph $G$ that is homotopic to the identity. Then the square of its rotation matrix is the zero matrix.

**Example 1 continued.** In Example 1, the the oriented Markov matrix is

$$R(L_f) = \frac{1}{6} \begin{pmatrix} -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

The columns show that edges $E_1$ and $E_5$ get mapped once around the circle in an orientation reversing way, and edges $E_1$ and $E_5$ get mapped...
in an orientation preserving way. The entries of \( \pm \frac{1}{6} \) along the main diagonal correspond to the width of the rotation interval.

**Example 2 continued.** In Example 2,

\[
R(L_f) = \frac{1}{6} \begin{pmatrix}
3 & -3 & 0 & 3 & -3 & 0 \\
3 & -3 & 0 & 3 & -3 & 0 \\
-2 & 2 & 0 & -2 & 2 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0
\end{pmatrix}.
\]

The non-zero columns show edges that are mapped three times around the circle \( E_1E_2E_3 \) and once around the circle \( E_3E_4E_5E_6 \). Since \( M^6(f)_{1,1} = 4 \). The map \( L^6_f \) has a 4-horseshoe within edge \( E_1 \). Thus the topological entropy of \( f \) is at least \( \frac{1}{6} \log 4 \).

For maps on trees there is no rotation, because there are no circles within the graph. Since every map on a tree is homotopic to the identity it is straightforward to show the following.

**Theorem 6.** Let \( f \) be a vertex map on a tree \( T \). Then \( R(L_f) \) is the zero matrix.

The next result shows that if the vertices form one periodic orbit then again the rotation matrix is the zero matrix.

**Theorem 7.** Let \( f \) be a vertex map on a graph \( G \) that is homotopic to the identity. If the vertices belong to one periodic orbit, then \( R(L_f) \) is the zero matrix.

**Proof.** Let \( m \) denote the period of the vertices. As above, if \( E_t \) is an edge in \( G \) that goes from vertex \( V_i \) to \( V_j \), then \( FC(L^m_f(E_t)) = FC((-P^m(V_i))E_tP^m(V_j)) \). Since \( \theta^m \) is the identity, both \( P^m(V_i) \) and \( P^m(V_j) \) must be closed loops in \( G \). Since \( V_i \) and \( V_j \) belong to the same orbit, both \( P^m(V_i) \) and \( P^m(V_j) \) correspond to exactly the same closed loop. Thus the number of times \( E_t \) appears in \( P^m(V_i) \) is exactly the same as \( P^m(V_j) \). So the number of times that \( E_t \) appears in \((-P^m(V_i))P^m(V_j) \) minus the number of times that \( -E_t \) appears in \((-P^m(V_i))P^m(V_j) \) is zero. So \( M(L_f)^m - I = 0 \).

We now turn attention to the cases when the rotation matrix is not necessarily the zero matrix.

**Lemma 6.** Let \( f \) be a vertex map on a graph \( G \) that is homotopic to the identity with vertex permutation \( \theta \in S_v \). Then the trace of \( R(L_f) \) is zero.
Figure 3. Graph with vertex permutation (1)(2 3)

Proof. It is shown in [4] that the trace of an oriented Markov graph is equal to the number of edges minus the number of vertices that are not fixed by the map. If $n$ is a positive integer such that $\theta^n$ is the identity, then the trace of $M(L^n f)$ equal the number of edges. This means that $M(L^n f) - I$ will have trace zero.

\[\square\]

An immediate consequence of the lemma above is the following.

**Theorem 8.** Let $f$ be a vertex map on a graph $G$ that is homotopic to the identity with vertex permutation $\theta$. If the rotation matrix, $R(L f)$ has a non-zero entry along the main diagonal then there exist horseshoes under some iterate of the map, and the topological entropy of $f$ is non-zero.

Proof. If the main diagonal contains a non-zero entry, then it must contain both a positive entry and a negative entry by the lemma above. If $m$ is the smallest positive integer such that $\theta^m$ is the identity then $M(L f)^m = I + mR(L f)$ and so $M(L f)^m$ must contain at least one entry along the main diagonal that is greater than one.

\[\square\]

For maps of the circle the existence of rotation intervals means that there is an interval that gets mapped around a circle by some power of the map and this implies the existence of horseshoes. However, in the general case of graphs it is possible to have an interval that gets mapped around a circle under an iterate of the map, but not have any horseshoes. The difference being that in the general case the circle contained within the graph need not contain the interval. We conclude with an example of a map that has a non-zero rotation matrix, but has entropy zero.
**Example 3.** Consider the example depicted in Figure 3. Here $V_1$ is fixed and the other two vertices form a periodic orbit of period 2. Since it is of degree one, $M(L_f) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and so $R(L_f) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

**References**


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