A SHARKOVSKY THEOREM FOR VERTEX MAPS ON TREES

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Abstract. Let $T$ be a tree with $n$ vertices. Let $f : T \to T$ be continuous and suppose that the $n$ vertices form a periodic orbit under $f$. We show:

1. If $n$ is not a divisor of $2^k$ then $f$ has a periodic point with period $2^k$.
2. If $n = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then $f$ has a periodic point with period $2^p r$ for any $r \geq q$.
3. The map $f$ also has periodic orbits of any period $m$ where $m$ can be obtained from $n$ by removing ones from the right of the binary expansion of $n$ and changing them to zeroes.

Conversely, given any $n$, there is a tree with $n$ vertices and a map $f$ such that the vertices form a periodic orbit and $f$ has no other periods apart from the ones given above.

1. Introduction

In this paper we consider maps on trees. We will denote a map associated to a tree as a vertex map if the vertices form exactly one periodic orbit. (Note that the trees we consider are trees in the combinatorial sense i.e. where it is possible to remove a vertex and divide the tree into two connected components as opposed to the topological sense where this is not possible). There have been a number of papers that have studied maps on trees. Baldwin began this study by considering certain types of trees (see [6]) and it has been continued by work of Alsedà working in conjunction with Guaschi, Juher, Llibre, Los, Misirurewicz, Manosas, Mumbru and Ye (see [1], [2], [3] and [4]). The study of vertex maps was started in [8]. We generalize Theorem 1 of that paper.

Given a periodic orbit on an interval, we can consider the periodic points to be the vertices on the convex hull of the orbit. In this way we can consider the study of periodic orbits on the interval as a special case as the study of vertex maps. Sharkovsky’s theorem [10] becomes

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a theorem about vertex maps in this special case when the tree is topologically an interval. In this paper we prove a Sharkovsky-type theorem for vertex maps in general.

The theorem makes use of a process in which the number $n$ is written in binary and then the rightmost one in its expansion is changed to a zero. The process is then repeated until it ends with the number zero. For example, 31 has binary expansion 11111. Applying the process to this number yields the following binary expansions 11110, 11100, 11000, 10000 and 00000, or in decimal notation 30, 28, 24, 16 and 0.

We now state the result of this paper:

**Theorem 1.** Let $T$ be a tree with $n$ vertices. Let $f : T \to T$ be continuous and suppose that the $n$ vertices form a periodic orbit under $f$. We show:

1. (a) If $n$ is not a divisor of $2^k$ then $f$ has a periodic point with period $2^k$.
   (b) If $n = 2^p q$, where $q > 1$ is odd and $p \geq 0$, then $f$ has a periodic point with period $2^p r$ for any $r \geq q$.
   (c) The map $f$ also has periodic orbits of any period $m$ where $m$ can be obtained from $n$ by removing ones from the right of the binary expansion of $n$ and changing them to zeroes.

2. Conversely, given any $n$, there is a tree with $n$ vertices and a vertex map $f$ that has no other periods apart from the ones given above.

We make the following observations. First, parts 1(a) and 1(b) were proved in [8]. If we restrict to the periods greater than $n$ then the set of periods given by Theorem 1 is exactly the same set of periods greater than $n$ that are forced by the standard Sharkovsky theorem on the interval. The difference between Sharkovsky’s theorem and Theorem 1 is in the set of periods forced that have period less than $n$. For example, in Sharkovsky’s theorem 31 forces all even integers less than 31, but Theorem 1 says that 31 forces only 24, 28 and 30 in addition to the powers of 2. The second part of the theorem shows that part 1 is best possible in that for any $n$ there exists a vertex map with exactly the set of periods given by part 1.

2. **Basics**

Given a connected tree $T$ with $n$ vertices let $V_1, \ldots, V_n$ denote the vertices and $E_1, \ldots, E_{n-1}$ denote the edges. Let $[V_i, V_j]$ denote the shortest path in the tree that connects $V_i$ to $V_j$.

Suppose $\theta$ is a permutation on $1, \ldots, n$. Then we define $L_{\theta}$, the connect-the-dot map, to be the continuous map from $T$ to itself given
by $L_\theta$ maps $[V_k, V_i]$ linearly onto $[V_{\theta(k)}, V_{\theta(i)}]$ for each edge $[V_k, V_i]$ in the tree.

We define the Markov Graph associated to the tree $T$ and $L_{\theta}$ in the following way: the vertices of the Markov graph correspond to the edges of the tree (we will abuse notation and also denote these vertices by $E_1, \ldots, E_{n-1}$, the context should make it clear whether $E_k$ refers to the edge in the tree or the corresponding vertex in the Markov graph); we draw a directed edge from vertex $E_i$ to vertex $E_j$ if and only if $L_{\theta}(E_i) \supseteq E_j$. We will denote this Markov graph by $G(\theta)$.

The next result is standard (see [9] or [5], for example, for a formal proof).

**Lemma 1.** Let $\theta \in S_n$. Suppose that $E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0}$ is a loop in $G(\theta)$. Then there exists a periodic point $x$ of $L_{\theta}$ with $L_{\theta}^{m+1}(x) = x$ such that $L_{\theta}^{r}(x) \in E_{k_r}$ for $r = 0, 1, \ldots, m$. Conversely, if $x$ is a periodic point in $L_{\theta}$ of period $m + 1$ and if $x$ is not a vertex then there exists a loop $E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0}$ in $G(\theta)$ such that $L_{\theta}^{r}(x) \in E_{k_r}$ for $r = 0, 1, \ldots, m$.

As in [8] we can extend the first half of this lemma to continuous maps.

**Lemma 2.** Suppose that $f$ is a continuous map on a tree with $n$ vertices labeled 1 through $n$. Let $\theta \in S_n$ be such that $f(V_i) = V_{\theta(i)}$ for each vertex. Suppose that $E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0}$ is a loop in $G(\theta)$. Then there exists a periodic point $x$ of $f$ with $f^{m+1}(x) = x$ such that $f^{r}(x) \in E_{k_r}$ for $r = 0, 1, \ldots, m$.

If $\theta$ consists of a single cycle and if the loop $E_{k_0} \rightarrow E_{k_1} \rightarrow \ldots E_{k_m} \rightarrow E_{k_0}$ in the above statement is not a repetition of a shorter loop then it can be checked that the period of $x$ is $m + 1$. In such a case we will say the that there is a non-repetitive loop of length $m + 1$.

To prove the first part of Theorem 1 it is enough to prove the following:

**Lemma 3.** Let $T$ be a tree and $f$ a continuous map from $T$ to itself. Suppose that the tree has $n$ vertices labeled 1 through $n$. Let $\theta \in S_n$ be such that $f(V_i) = V_{\theta(i)}$ for each vertex. Suppose that $\theta$ consists of a single cycle of length $n$.

1. If $n$ is not a divisor of $2^k$ then $G(\theta)$ has a non-repetitive loop of length $2^k$.
2. If $n = 2^pq$, where $q > 1$ is odd and $p \geq 0$. Then $G(\theta)$ has a non-repetitive loop of length $2^pr$ for any $r \geq q$. 


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(3) $G(\theta)$ also has non-repetitive loops of any length $m$ where $m$ can be obtained from $n$ by removing ones from the right of the binary expansion of $n$ and changing them to zeroes.

We now choose an orientation for each of the edges and define the oriented Markov Graph associated to the tree $T$ and $L_\theta$ to be the Markov graph defined above with the addition of plus or minus signs to each of the directed edges to keep track of orientation. More formally, suppose there is a directed edge from vertex $E_i$ to vertex $E_j$. We assign a sign of $+$ or $-$ to this directed edge according to whether $L_\theta$ maps $E_i$ onto $E_j$ in an orientation preserving or orientation reversing way.

Let $T$ be a fixed tree with $n$ vertices. Assigned to each $\theta \in S_n$ is an oriented transition matrix, denoted $M(\theta)$ and defined by

$$M(\theta)_{i,j} = \begin{cases} 
1 & \text{if there is a positive edge from } E_j \text{ to } E_i, \\
-1 & \text{if there is a negative edge from } E_j \text{ to } E_i, \\
0 & \text{otherwise}.
\end{cases}$$

In [8] the following results were proved.

**Lemma 4.** Let $\theta \in S_n$. Then the $ij$th entry of $[M(\theta)]^k$ equals the number of orientation preserving paths from $E_i$ to $E_j$ of length $k$ minus the number of orientation reversing paths of length $k$.

**Theorem 2.** Let $\theta \in S_n$. If $\theta(i) \neq i$ for $1 \leq i \leq n$ then $\text{Trace}(M(\theta)) = -1$.

**Theorem 3.** Suppose that $\alpha, \beta \in S_n$. Then $M(\alpha)M(\beta) = M(\alpha\beta)$.

3. Proof of Lemma 3

As noted above to prove the first part of Theorem 1 it is sufficient to prove Lemma 3. The proof of part 1 of Lemma 3 was given in [8] we repeat it here because it is short and aids the exposition.

3.1. Proof of Lemma 3 part (1).

**Proof.** We begin by considering $\theta \in S_n$ where $n$ does not divide $2^k$. Note that $\theta^{2^j}$ does not fix any of $1, \ldots, n$ for $1 \leq j \leq k$, so by Theorem 2 we know that $\text{Trace}(M(\theta^{2^j})) = -1$. Theorem 3 then tells us that $\text{Trace}(M(\theta^{2^j})) = \text{Trace}((M(\theta))^{2^j}) = -1$ for $1 \leq j \leq k$. Lemma 4 shows that there must be at least one orientation reversing path from a vertex $E$ to itself in $G(\theta)$ of length $2^j$ for $1 \leq j \leq k$. This loop must be non-repetitive because any repetitive loop of length $2^j$ would have to be an even repetition of a shorter loop and thus have positive orientation. \qed
3.2. Some technical lemmas. Let $\theta \in S_n$. We only need to consider the cases when $n$ is not a power of 2. In these cases $n$ does not divide $2^k$ for any $k \geq 0$ and so the first part of Lemma 3 tells that there must be a non-repetitive loop of length $2^k$. The proof above shows that we can strengthen this statement a little more and say that there must be an orientation reversing non-repetitive loop of length $2^k$ for any $k \geq 0$. The rest of the proof of Lemma 3 is obtained by looking at these loops and seeing when they must intersect other loops.

It is here that the proof of Theorem 1 differs from the proof of Sharkovsky’s Theorem. When the tree is topologically an interval it is the case that if $f^k$ fixes more than one edge then $f^k$ contains a 2-horseshoe and so must $f^k$ must have periodic orbits of all periods. For trees in general it is possible for $f^k$ to have many fixed edges but still not contain a horseshoe. For more information about this and the proof of Sharkovsky’s theorem see [7].

Lemma 5. Let $\theta \in S_n$ where $n$ is not a power of 2. For a positive integer $m$ consider the non-repetitive loops of length $2^j$ in $G(\theta)$ for $0 \leq j \leq m$. If these loops are not disjoint then $G(\theta)$ has non repetitive loops of length $k2^m$ for any $k \geq 1$.

Proof. Suppose non-repetitive loops of length $2^t$ and $2^s$ intersect. Then we can obtain a non-repetitive loop of length $k2^m$ by going $2^{m-t}$ times around the loop of length $2^t$ and $2^{m-s}$ times around the loop of length $2^s$. $\square$

We will start removing edges from the tree $T$. (When we remove an edge we leave the vertices that consist of the endpoints of the edges.) Suppose a tree has $s$ vertices. Removing an edge from this tree results in two trees. Consider the tree with the least number of vertices. If $s$ is even this number can be at most $s/2$, if $s$ is odd this number can be at most $(s-1)/2$. We define $h : \mathbb{N} \to \mathbb{N}$ by $h(s) = s/2$ if $s$ is even and $h(s) = (s-1)/2$ if $s$ is odd. We will let $h^n$ denote the $n$th iterate of $h$ and $h^0$ denote the identity map.

We now consider the case when the loops of length $2^k$ for $0 \leq k \leq j$ are disjoint. The matrix $M(\theta)^{2^j}$ will have at least $2^k$ entries of +1 along the main diagonal corresponding to the loop(s) of length $2^k$ for $0 \leq k < j$, and at least $2^j$ entries of −1 along the main diagonal corresponding to the loop(s) of length $2^j$. This means that we must have at least $2^j - 1$ entries that are equal to +1. Consider the edges in the tree that correspond to these entries. We will denote these edges as expanding edges under $L_{2^j}^{\theta}$. Note that applying $L_{2^j}^{\theta}$ to an expanding edge gives a path which contains the expanding edge as an interior
edge, so removing an expanding edge never gives an isolated vertex. The following is easily proved by induction

**Lemma 6.** Let $\theta \in S_n$ where $n$ is not a power of 2. Suppose the non-repetitive loops in $G(\theta)$ of length $2^k$ for $0 \leq k < j$ are disjoint. Suppose the expanding edges under $L_{\theta}^{2^j}$ are removed from the tree $T$. Then the result is at least $2^j$ trees. The tree with the least number of vertices will have at most $h^j(n)$ vertices.

**Lemma 7.** Let $\theta \in S_n$ where $n$ is not a power of 2. Suppose the non-repetitive loops in $G(\theta)$ of length $2^k$ for $0 \leq k < j$ are disjoint. Suppose $h^{j-1}(n)$ is odd. Then $G(\theta)$ contains non-repetitive loops of length $k2^j h^j(n) + l2^{j-1}$ for any integers $k > 0$ and $l \geq 0$.

**Proof.** As above there are at least $2^j - 1$ edges in $T$ that are expanding under $L_{\theta}^{2^j}$. Each one of these edges corresponds to a non-repetitive loop of length $2^k$ for $0 \leq k < j$ and so each corresponds to a possibly repetitive loop of length $2^{j-1}$. We consider two cases.

In the first case suppose that one of these expanding edges $E_i$ ceases to be expanding under $L_{\theta}^{2^j r}$ for $0 \leq r \leq h^j(n)$. Let $s$ denote the smallest positive integer such $E_i$ is not expanding under $L_{\theta}^{2^j s}$. This means that the $i$th entry on the main diagonal of $M(\theta)^{2^j r}$ equals 1 for $1 \leq r < s$ but is not equal to 1 when $r = s$. This implies that in addition to the positive (repetitive) loop of length $2^{j-1} 2s$ that this edge belongs to there must be another negative loop of length $2^j s$. Since $s$ is minimal this loop must be non-repetitive. We can obtain a non-repetitive loop of length $k2^j h^j(n) + l2^{j-1}$ by first going around the negative loop $k$ times and going $2k(h^j(n) - s) + l$ times around the positive loop of length $2^{j-1}$.

In the second case we consider what happens when all the expanding edges remain expanding under $L_{\theta}^{2^j r}$ for $0 \leq r \leq h^j(n)$.

As in the previous lemma removing the expanding edges results in at least $2^j$ trees. We will let the tree with the least number of vertices be denoted by $T^*$. The previous lemma shows that $T^*$ has at most $h^j(n)$ vertices.

We now add back the expanding edges and consider $T^*$ within $T$. The only edges that connect a vertex in $T^*$ to a vertex not in $T^*$ are expanding edges and each one of these edges corresponds to a non-repetitive loop of length $2^k$ for $0 \leq k < j$ and so each corresponds to a possibly repetitive loop of length $2^{j-1}$. It is enough to complete the proof if we can show that one of these edges is also on a distinct loop of length $2^j h^j(n)$. 

Suppose that there are \( e \) of these expanding edges connecting \( T^* \) to the rest of \( T \). We will denote these edges by \( E^*_1 \ldots E^*_e \) and the vertices that both belong to \( T^* \) and an expanding edge by \( V_1^*, \ldots, V_{e}^* \). Since, by hypothesis, \( h^{j-1}(n) \) is odd we know that \( 2^jh^j(n) < n \) and so \( L^{2^jh^j}(V_i^*) \neq V_i^* \) for \( 0 \leq w \leq h^j(n) \). Since \( T^* \) contains at most \( h^j(n) \) vertices it must be the case that \( L^{2^jw}(V_i^*) \) will map any vertex in \( T^* \) to some vertex that does not lie in \( T^* \) for at least one value of \( w \) satisfying \( 0 \leq w \leq h^j(n) \).

Let \( k_i \) for \( 1 \leq i \leq e \) denote the smallest positive integer such that \( L^{2^jk_i}(V_i^*) \not\in T^* \). Notice that \( L^{2^jk_i}(E_i^*) \) must contain another of the expanding edges that connect \( T^* \) to \( T \) because \( L^{2^jk_i}(E_i^*) \) is expanding on \( E_i \) and the only paths to the other vertices of \( T \) are through the other expanding edges. Thus in \( G(\theta) \) there is a path from \( E_i^* \) to one of the other \( E^* \) edges of length \( k_i \). Since this statement is true for \( 1 \leq i \leq e \) we must be able to find a loop \( E_{q_0}^* \rightarrow E_{q_1}^* \rightarrow \ldots E_{q_m}^* \rightarrow E_{q_0}^* \), where the arrows denote paths of length \( 2^kk_{q_0}, 2^jk_{q_1}, \ldots 2^kk_{q_m} \) in \( G(\theta) \). We will assume that \( E_{q_0}^* \rightarrow E_{q_1}^* \rightarrow \ldots E_{q_m}^* \rightarrow E_{q_0}^* \) is the loop of least length that can be constructed in this way i.e. such that \( k_{q_0} + k_{q_1} + \cdots + k_{q_m} \) is a minimum.

Now for \( 0 \leq i \leq m \) consider the vertices \( V_{q_i}^* \) that belongs to both \( E_{q_i}^* \) and to the tree \( T^* \). For \( 0 \leq w < k_{q_i} \) we have \( L^{2^jw}(V_{q_i}^*) \in T^* \). Since we are choosing a loop of minimum length it must be the case that \( \{L^{2^jw}(V_{q_i}^*)|0 \leq w < k_{q_i}\} \) and \( \{L^{2^jw}(V_{q_u}^*)|0 \leq w < k_{q_u}\} \) are disjoint if \( i \neq u \). Since \( T^* \) has at most \( h^j(n) \) vertices we must have \( k_{q_0} + k_{q_1} + \cdots k_{q_m} = z \leq h^j(n) \) and so the loop from \( E_{q_0}^* \) to itself has length \( 2^jz \).

We can now obtain a non-repetitive loop of length \( k2^jh^j(n) + l2^{j-1} \) by going \( k \) times around the loop of length \( 2^jh^j \) and \( k2(h^j(n) - z) + l \) times around the loop of length \( 2^{j-1} \).

Our final observation is that if \( h^{j-1}(n) \) is even then \( 2h^j(n) = h^{j-1}(n) \) and so \( 2^jh^j(n) = 2^{j-1}h^{j-1}(n) \). This observation and the previous lemma give us:

**Lemma 8.** Let \( \theta \in S_n \) where \( n \) is not a power of 2. Suppose the non-repetitive loops in \( G(\theta) \) of length \( 2^k \) for \( 0 \leq k < j \) are disjoint. Then \( G(\theta) \) contains non-repetitive loops of length \( 2^ih^i(n) \) for \( 1 \leq i \leq j \).

### 3.3. Proof of part (2) of Lemma 3.

**Proof.** Since \( n = 2^pq \) where \( q > 1 \) is odd part (1) of Lemma 3 tells us that \( G(\theta) \) must have non-repetitive loops of length \( 2^j \) for \( 1 \leq j \leq p \). If these loops are not disjoint then Lemma 5 completes the proof. If they are disjoint then \( h^p(n) = q \) is odd and \( 2h^{p+1} < q \). Lemma 7 tells us
that there must be non-repetitive loops of length $k2^{p+1}h^{p+1}(n) + l2^p$. Taking $k = 1$ and $l = r - 2h^{p+1}(n)$ gives the required result.

3.4. Proof of part (3) of Lemma 3.

Proof. The process of taking $n$ and then removing ones from the right results in a set of positive even integers that lie between $n/2$ and $n$. Notice that this set of integers is exactly \{2^i h^i(n) | 1 \leq i \leq \log_2(n)\}. Consider the non-repetitive loops of length $2^j$ for $1 \leq j \leq p$, we know they exist by part (1) of Lemma 3. If these are all disjoint then Lemma 8 completes the proof. If they are not all disjoint let $t$ denote the largest positive integer such that non-repetitive loops of length $2^j$ for $1 \leq j \leq t$ are disjoint. Lemma 8 shows that we have non-repetitive loops of length $2^t h^t(n)$ for $1 \leq i \leq t + 1$. Lemma 5 shows that we have non-repetitive loops of length $k2^{r+1}$ for any for any $k \geq 1$, so we must have non-repetitive loops of length $2^t h^t(n)$ for $i > t$.

4. Proof of part (2) of Theorem 1

To complete the proof of Theorem 1 we need to show that for any $n$ there is a tree and a vertex map that has exactly the periods given by part (1) of the theorem.

The required family and result is contained in [8]. We simply re-state these facts here.

We define a family of trees, $T_n$, recursively on the number of vertices $n$. If $n = 1$ there is just one possible tree and labeling, a single vertex labeled 1. Suppose that the tree and labeling has been defined for $2^m$ vertices. For $k$ satisfying $2^m < k \leq 2^{m+1}$ define a new edge that goes from vertex $k$ to vertex $k - 2^m$.

Given the tree $T_n$ we choose the map to be $L_\theta$ where $\theta$ is the permutation $(123 \ldots n)$.

The following is Theorem 6 in [8]. Notice that the periods are exactly the periods given in Theorem 1.

**Theorem 4.** Let $T_n$ be as defined above.

If $n = 2^m$ then $T_n$ has periodic points with periods $2^k$ for each $k$ satisfying $2^k \leq n$ and there are no others.

If $n = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_j}$ where $r_i < r_{i+1}$ and $j > 1$ then $T_n$ has periodic points with periods $2^k$ for each $k$, periods $n + v2^{r_1}$ for any $v$, periods $\sum_{t=m}^{2^j} 2^t$ for $m \geq 1$, and there are no others.

5. The partial order on the positive integers

We conclude the paper by showing that Theorem 1 gives a partial order on the positive integers and give a diagram showing this ordering.
We will say that an integer $a > 0$ implies integer $b > 0$ if

1. $a = b$, or
2. if $b = 2^k$ for some $k$ and $a$ is not a divisor of $b$, or
3. if $a = 2^p q$, where $q > 1$ is odd and $p \geq 0$ and $b = 2^r r$ for some $r > q$, or
4. if $b$ can be obtained from $a$ by removing ones from the right of the binary expansion of $a$.

We will say that an integer $a > 0$ forces integer $b > 0$ if there exists a finite sequence of positive integers $a = a_0, a_1, \ldots, a_n = b$ such that $a_i$ implies $a_{i+1}$ for $0 \leq i \leq n - 1$.

**Theorem 5.** The relation $a$ forces $b$ gives a partial order on the set of positive integers.

**Proof.** The only non-trivial fact to check is that if $a$ forces $b$ and $b$ forces $a$ then $a = b$.

First, note that the only integers implied by $2^n$ are integers of the form $2^m$ where $m \leq n$. So if either of $a$ or $b$ equals $2^n$ and $a$ forces $b$ and $b$ forces $a$ we must have $a = b = 2^n$.

We only need to consider the case when neither $a$ nor $b$ is a power of two. Notice this means that if $a \neq b$ the sequence of implications that take us from $a$ to $b$ to show $a$ forces $b$ will only be of the form given by (3) and (4) in the definition above. The proof will be complete if we can show that there is not a sequence of implications that takes us from $a$ to itself of type (3) and (4). Now if $a_i = 2^p q$ where $q$ is odd and if $a_{i+1}$ is obtained by (3) we must have $a_{i+1} = 2^t s$ where $t$ is odd and $s \geq p$. However, if $a_i = 2^p q$ where $q$ is odd and if $a_{i+1}$ is obtained by (4) we must have $a_{i+1} = 2^t s$ where $t$ is odd and $s$ is strictly greater than $p$. Thus as we use a sequence of operations of type (3) and (4) the powers of two in the prime factorizations increases whenever we use (3) and can never decrease when we use (4). So there cannot be a sequence of implications that takes us from $a$ to itself that uses any of type (4). However, if $a_i$ implies $a_{i+1}$ by (3) we have $a_{i+1} > a_i$. So there cannot be a sequence of type (4) implications from $a$ to itself.

Finally, we include a diagram to illustrate this ordering. The vertical columns denote the positive odd integers, two times the positive integers and so on. The ordering among the columns is the same as in the standard Sharkovsky ordering. However, the difference between the tree ordering in this paper and Sharkovsky’s ordering is given by the horizontal arrows which come from removing ones from the right.
Figure 1. Partial order on the positive integers

References


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