Math 231 HW #8

Section 5.3 9, 17, 26

5.4 2, 5

5.3 #9

Thm $6 \mid 7^n - 1 \quad \forall n \in \mathbb{Z}^{\text{non neg}}$

Pf by induction:

Base case ($n=0$): we need $6 \mid 7^0 - 1$, which means $6 \mid 0$ which is true since $0 \cdot 6 = 0$.

Inductive case: Assume $6 \mid 7^k - 1$, we show $6 \mid 7^{k+1} - 1$.

Since $6 \mid 7^k - 1$, let $7^k - 1 = 6a$ with $a \in \mathbb{Z}$:

Then
$$7(7^k - 1) = 42a$$
$$7^{k+1} - 7 = 42a$$
$$7^{k+1} - 1 = 42a + 6$$
$$7^{k+1} - 1 = 6(7a + 1)$$
so $6 \mid 7^{k+1} - 1$ as desired.

5.3 #17 Thm $1 + 3n \le 4^n$ for all $n \in \mathbb{Z}^{\text{non neg}}$

Pf by induction.

Base case ($n=0$): We need $1 + 3 \cdot 0 \le 4^0$ which is clear.

Inductive case: Assume $1 + 3k \le 4^k$, we'll show $1 + 3(k+1) \le 4^{k+1}$.

We have $1 + 3(k+1) = 1 + 3k + 3 \le 4^k + 3$ by IH
$$\le 4^k + 3 \cdot 4^k \le 4 \cdot 4^k = 4^{k+1}$$

So $1 + 3(k+1) \le 4^{k+1}$ as desired.

5.3 #26 $c_0 = 3$, $c_k = (c_{k-1})^2$ for $k \geq 1$.

Show $c_n = 3^{2^n}$ $\forall n \geq 0$.

Pf by induction.

Base case $n=0$: We need $c_0 = 3^{2^0} = 3$ which was given.

Inductive case Assume $c_k = 3^{2^k}$, will show $c_{k+1} = 3^{2^{k+1}}$.

We have $c_{k+1} = (c_k)^2 = (3^{2^k})^2 = 3^{2 \cdot 2^k} = 3^{2^{k+1}}$ as desired.

5.4 #2 $b_1 = 4$, $b_2 = 12$, $b_k = b_{k-2} + b_{k-1}$

Show $4 | b_n$ $\forall n \in \mathbb{Z}^+$

Pf by strong induction:

Base cases: $n=1$: We need $4|b_1$, but $b_1 = 4$ so this is clear

$n=2$: We need $4|12$, also clear.

Inductive case: Assume $4|b_i$ for all $i \leq k$, will show $4 | b_{k+1}$.

$b_{k+1} = b_{k-1} + b_k$, by IH $\exists \ell, m \in \mathbb{Z}$ s.t. $b_{k-1} = 4\ell$, $b_{k-2} = 4m$.

so $b_{k+1} = 4\ell + 4m = 4(\ell + m)$ so $4 | b_{k+1}$ as desired.

5.4 #5 $e_0 = 12$, $e_1 = 29$ $e_k = 5e_{k-1} - 6e_{k-2}$ for $k \geq 2$

Show $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ $\forall n \in \mathbb{Z}^{\text{nonneg}}$

Pf by strong induction.

Base cases $n=0$: need $12 = 5 \cdot 3^0 + 7 \cdot 2^0$ which is clear.

$n=1$: need $29 = 5 \cdot 3 + 7 \cdot 2$, again clear.

Induction Assume $e_i = 5 \cdot 3^i + 7 \cdot 2^i$ for $i \leq k$.

Then $e_{k+1} = 5e_k - 6e_{k-1} = 5(5 \cdot 3^k + 7 \cdot 2^k) - 6(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1})$

$= 25 \cdot 3^k - 30 \cdot 3^{k-1} + 35 \cdot 2^k - 42 \cdot 2^{k-1}$

$= 3^{k-1}(25 \cdot 3 - 30) + 2^{k-1}(35 \cdot 2 - 42)$

$= 3^{k-1} \cdot 45 + 2^{k-1} \cdot 28 = 3^{k-1} \cdot 3^2 \cdot 5 + 2^{k-1} \cdot 2^2 \cdot 7$

$= 3^{k+1} \cdot 5 + 2^{k+1} \cdot 7$ as desired.