A short sample $\LaTeX$ document

Chris Staecker

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Mendelson Exercise 2.2.3

We will show that
$$d(x,y) \le d'(x,y) \le \sqrt{n}d(x,y),$$
where d is the "maximum" metric on $\mathbb{R}^n$ and d' is the Euclidean metric on $\mathbb{R}^n$.

Throughout, we'll use x and y in coordinates:
$$x = (x_1, \dots, x_n),$$
$$y = (y_1, \dots, y_n).$$

First we'll show that $d(x,y) \le d'(x,y)$. We have:
$$d(x,y) = \max_{1 \le i \le n} |x_i - y_i|. \tag{1}$$

Choose some particular j such that $d(x,y) = |x_j - y_j|$. Then we have
$$d(x,y) = |x_j - y_j| = \sqrt{|x_j - y_j|^2} \le \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2} = d'(x,y)$$
as desired.

Now we'll show that $d'(x,y) \le \sqrt{n}\, d(x,y)$. Again, by (1), choose j such that $d(x,y) = |x_j - y_j|$. Then we have
$$\begin{aligned} d'(x,y) &= \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2} \le \sqrt{|x_j - y_j|^2 + \cdots + |x_j - y_j|^2} \\ &= \sqrt{n|x_j - y_j|^2} = \sqrt{n}|x_j - y_j| = \sqrt{n}d(x,y) \end{aligned}$$
as desired.

Mendelson Exercise 2.3.3

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x_1, x_2) = x_1 + x_2$. First we will show that f is continuous with respect to the "max" metric d:

Let $\epsilon > 0$ be given. We will find some $\delta > 0$ such that
$$\max(|x_1 - y_1|, |x_2 - y_2|) < \delta \implies |f(x_1, x_2) - f(y_1, y_2)| < \epsilon,$$

which is equivalent to

$$\max(|x_1 - y_1|, |x_2 - y_2|) < \delta \implies |x_1 + x_2 - (y_1 + y_2)| < \epsilon.$$

Note that it is enough to make $|x_1 - y_1| + |x_2 - y_2| < \epsilon$, since

$$|x_1 + x_2 - (y_1 + y_2)| = |x_1 - y_1 + x_2 - y_2| \leq |x_1 - y_1| + |x_2 - y_2|$$

by the triangle inequality.

Now we let $\delta = \epsilon/2$, and then if the max of $|x_1 - y_1|$ and $|x_2 - y_2|$ is less than δ, we have

$$|x_i - y_i| < \epsilon/2$$

for both $i = 1$ and $i = 2$. Then we have

$$|x_1 + x_2 - (y_1 + y_2)| \leq |x_1 - y_1| + |x_2 - y_2| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as desired.

Now we will show that f is continuous with respect to the Euclidean metric d'. To avoid doing all the work for this, we'll use a result of the above exercise, namely that

$$d(x, y) \leq d'(x, y).$$

Let $\epsilon > 0$ be given, then we must find $\delta > 0$ such that

$$d'((x_1, x_2), (y_1, y_2)) < \delta \implies |f(x_1, x_2) - f(y_1, y_2)| < \epsilon.$$

But if $d'((x_1, x_2), (y_1, y_2)) < \delta$, we have

$$d((x_1, x_2), (y_1, y_2)) < \delta$$

automatically. So really we need only find δ so that

$$d((x_1, x_2), (y_1, y_2)) < \delta \implies |f(x_1, x_2) - f(y_1, y_2)| < \epsilon.$$

But this is exactly what we have already done above in showing continuity of f with respect to d, and so we are finished.