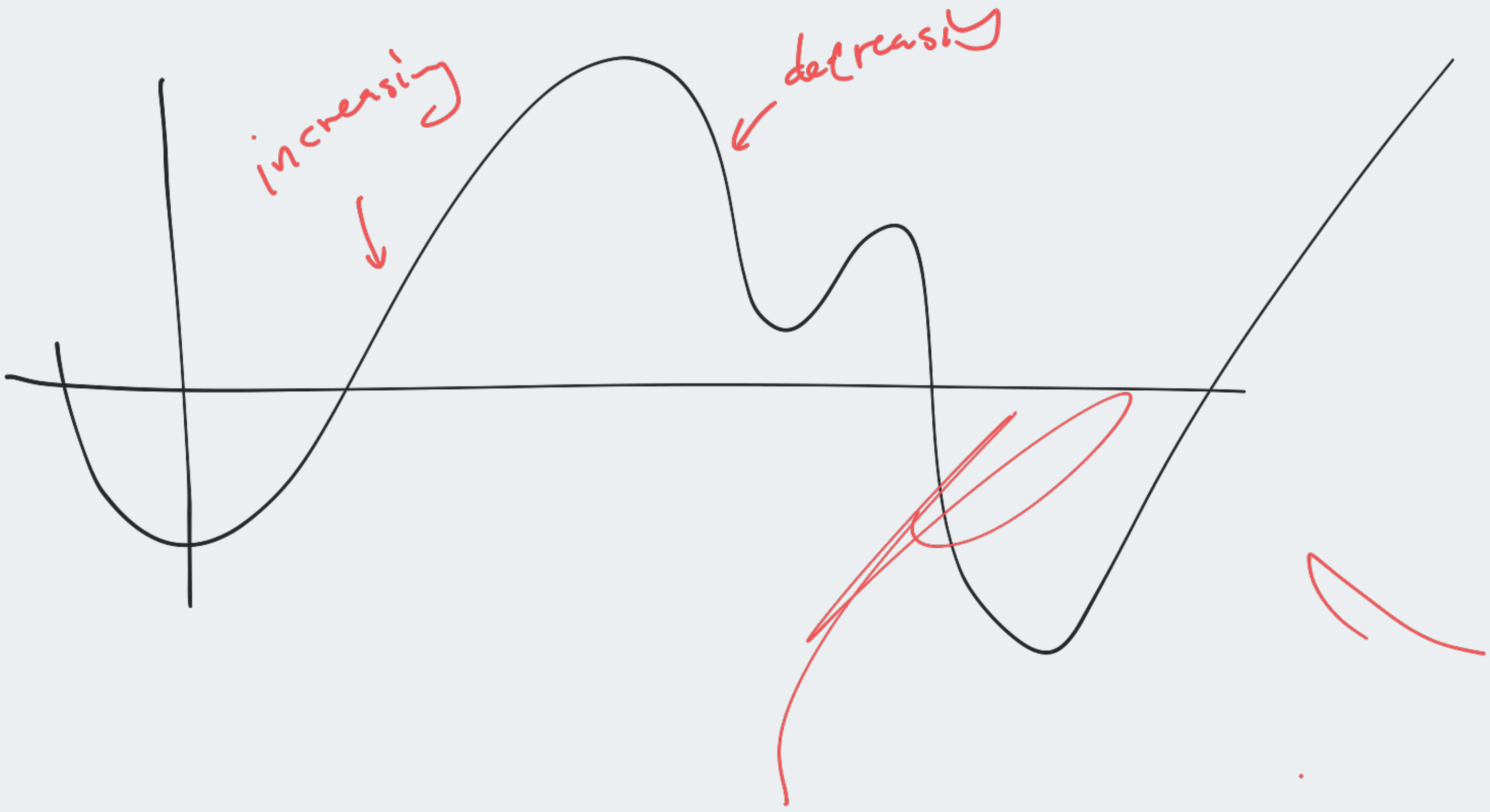
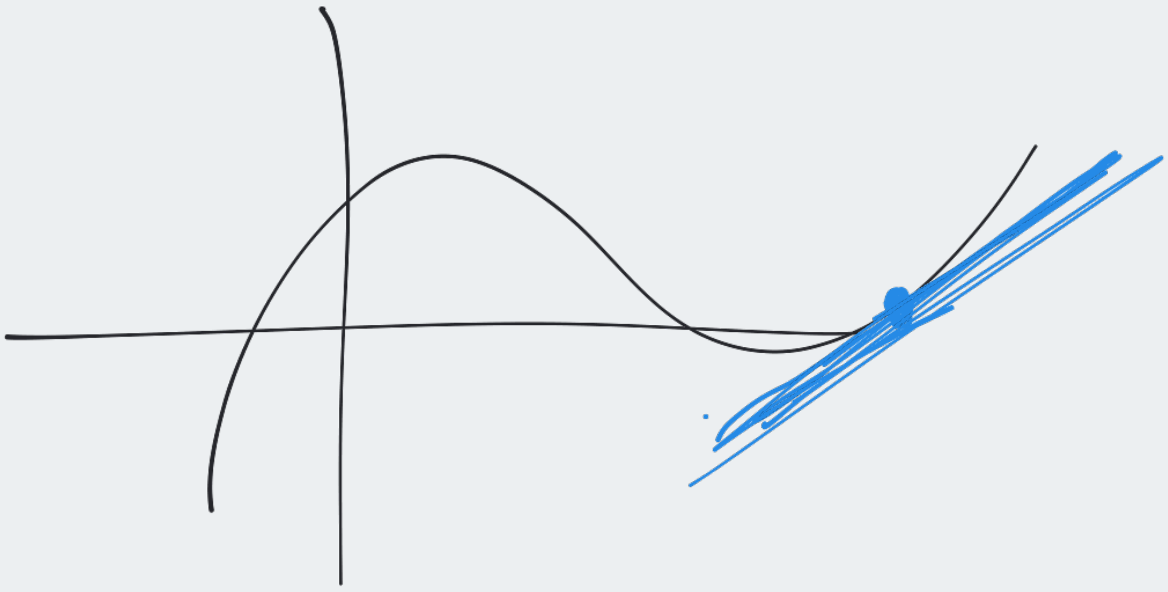
Slope & Tangents



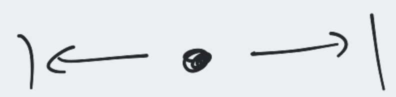
To measure slope of a curve,
best we can do is the slope of the <u>tangent</u> <u>line</u>

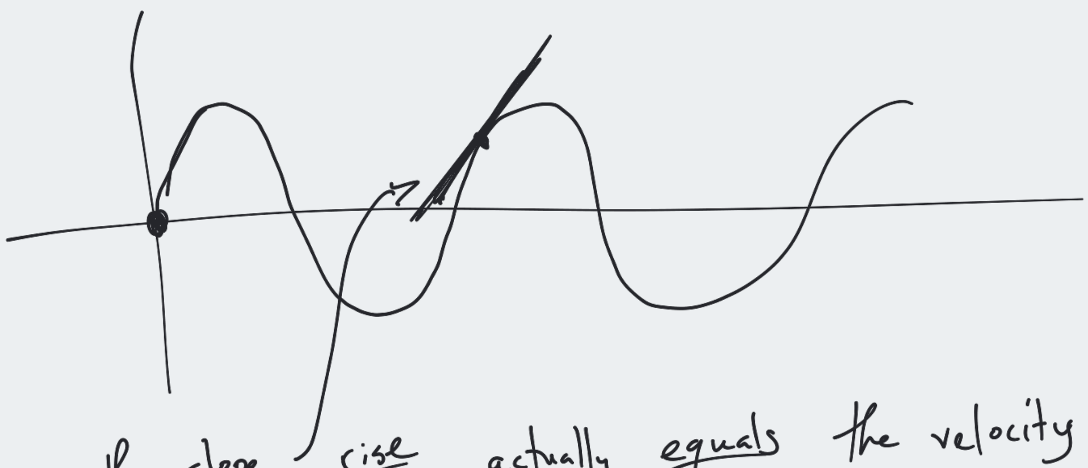


Super-useful in physics

Say $s(t)$ is the position of an object, in meters after time t seconds

Like a point moving back & forth





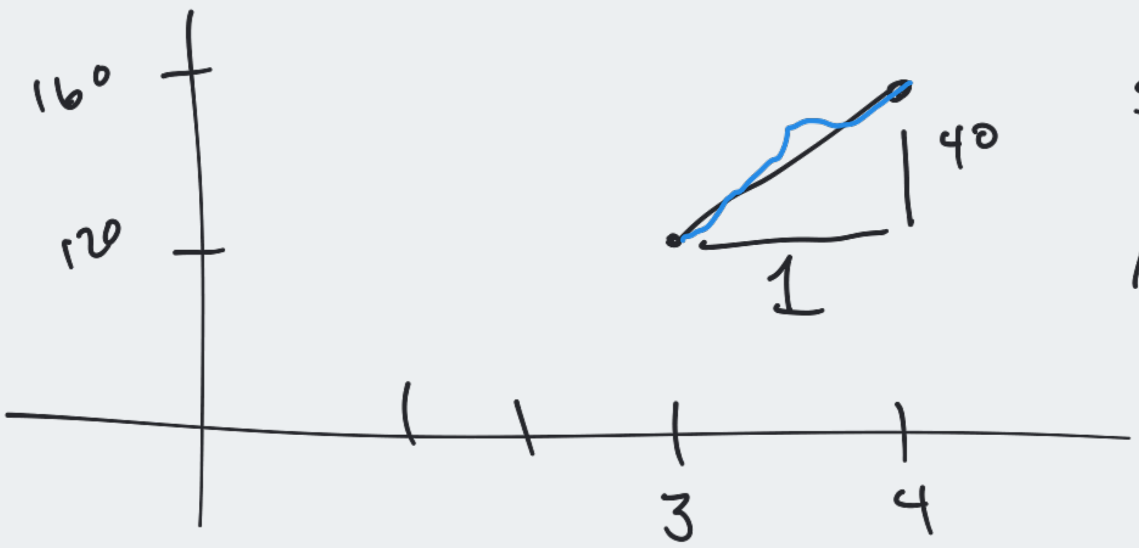
the slope $\frac{\text{rise}}{\text{run}}$ actually equals the velocity in meters/second

The velocity = slope of the tangent line.

Kinda obvious:

I'm driving, after 3 hrs, I've gone 120 miles
after 4 . - - - - - - 160 miles.

How fast was I going? 40 mph



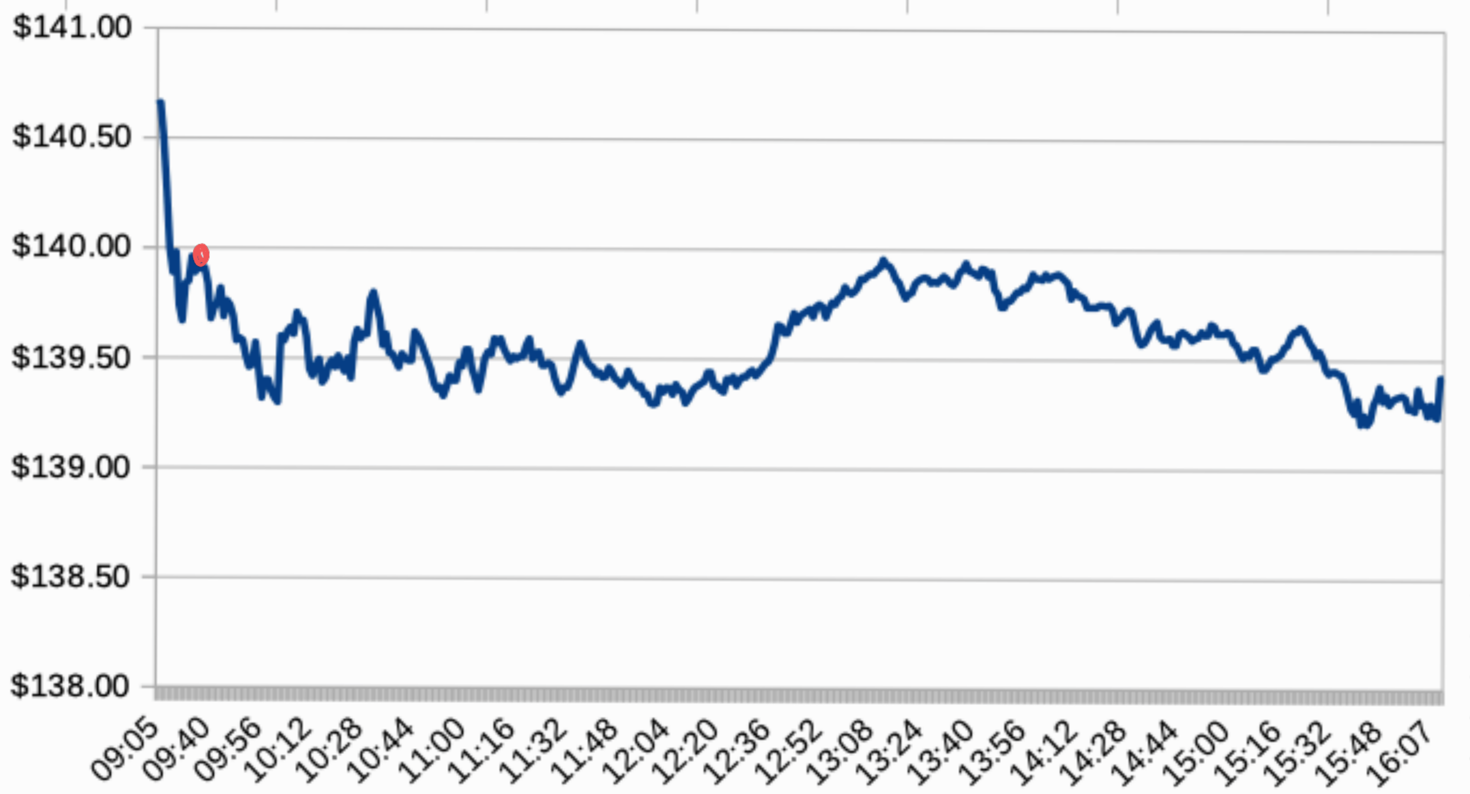
slope here is 40

Actually this slope is my average speed over the hour.

With real data:

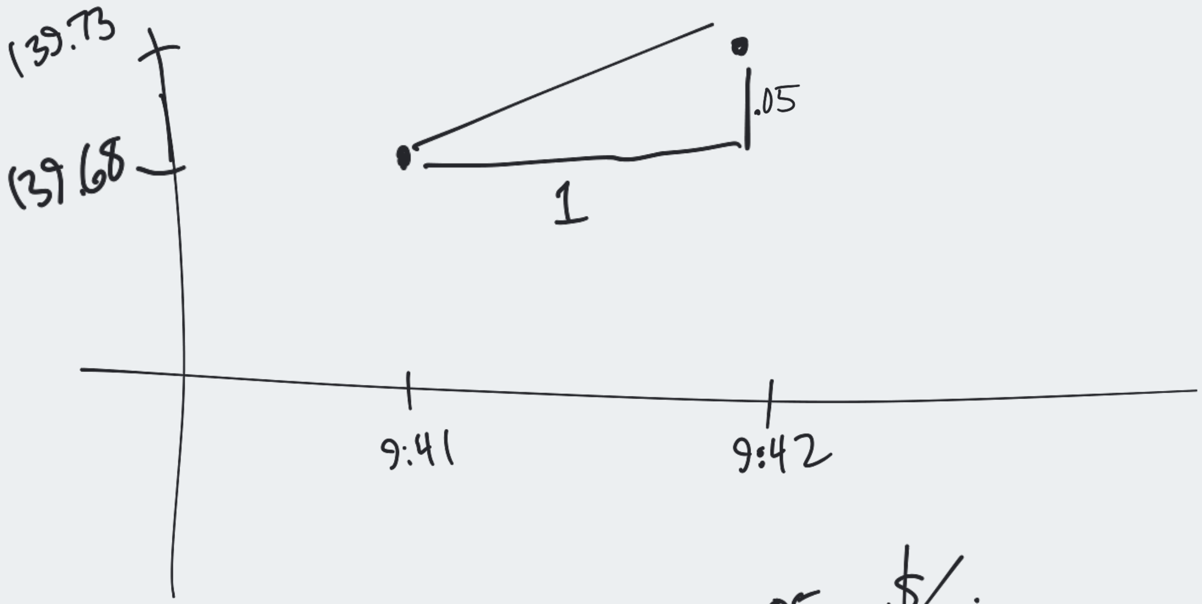
Time	Price
09:02	$140.38
09:05	$140.65
09:07	$140.66
09:13	$140.50
09:20	$140.27
09:28	$140.00
09:29	$139.89
09:30	$139.98
09:31	$139.74
09:32	$139.67
09:33	$139.84
09:34	$139.85
09:35	$139.96
09:36	$139.89
09:37	$139.95
09:38	$139.91
09:39	$139.91
09:40	$139.84
09:41	$139.68
09:42	$139.73
09:43	$139.75
09:44	$139.82
09:45	$139.69
09:46	$139.76
09:47	$139.74

IBM stock prices on September 9, 2021



How fast was it changing ($/min) at 9:41?

zooming in at 9:41, looks like



change was .05 $/min

Major line of inquiry:

If I know $f(x)$, how do I find the slope of the tangent at a particular point?

"The tangent problem"

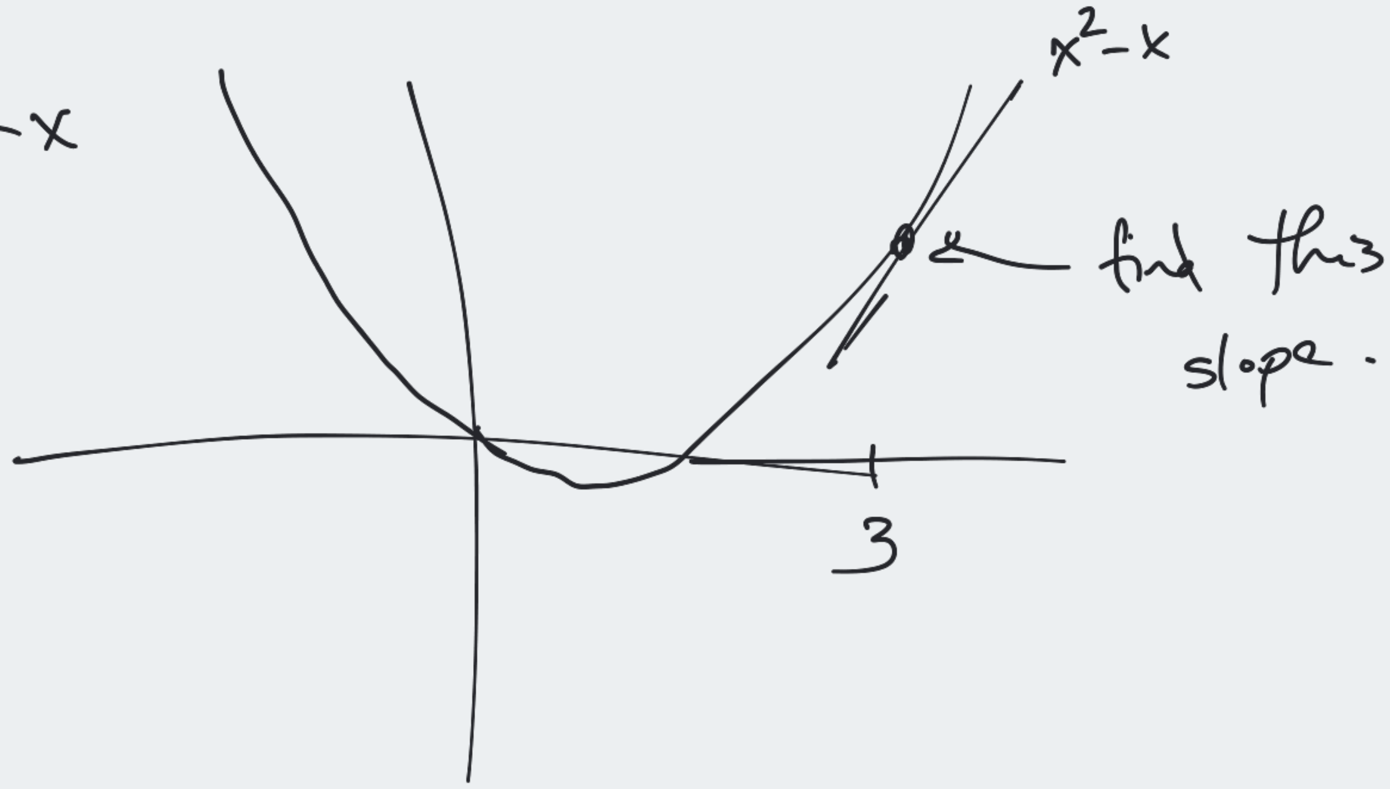
at least since Euclid (~300 BC)

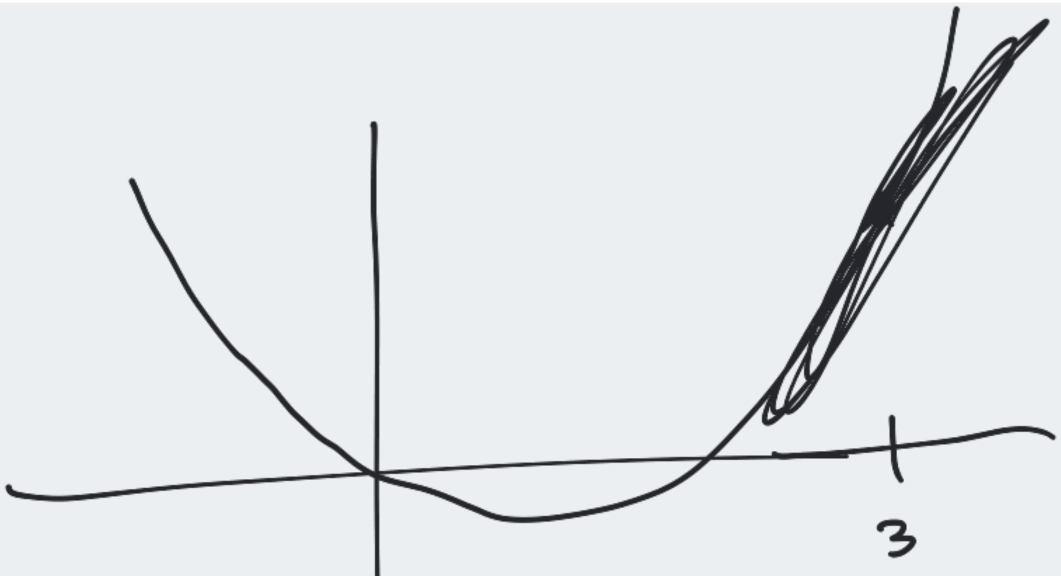
Solved by Newton & Leibniz

figured a general method for any function.

<u>Basic idea:</u> Consider smaller & smaller time intervals between points.

$f(x) = x^2 - x$





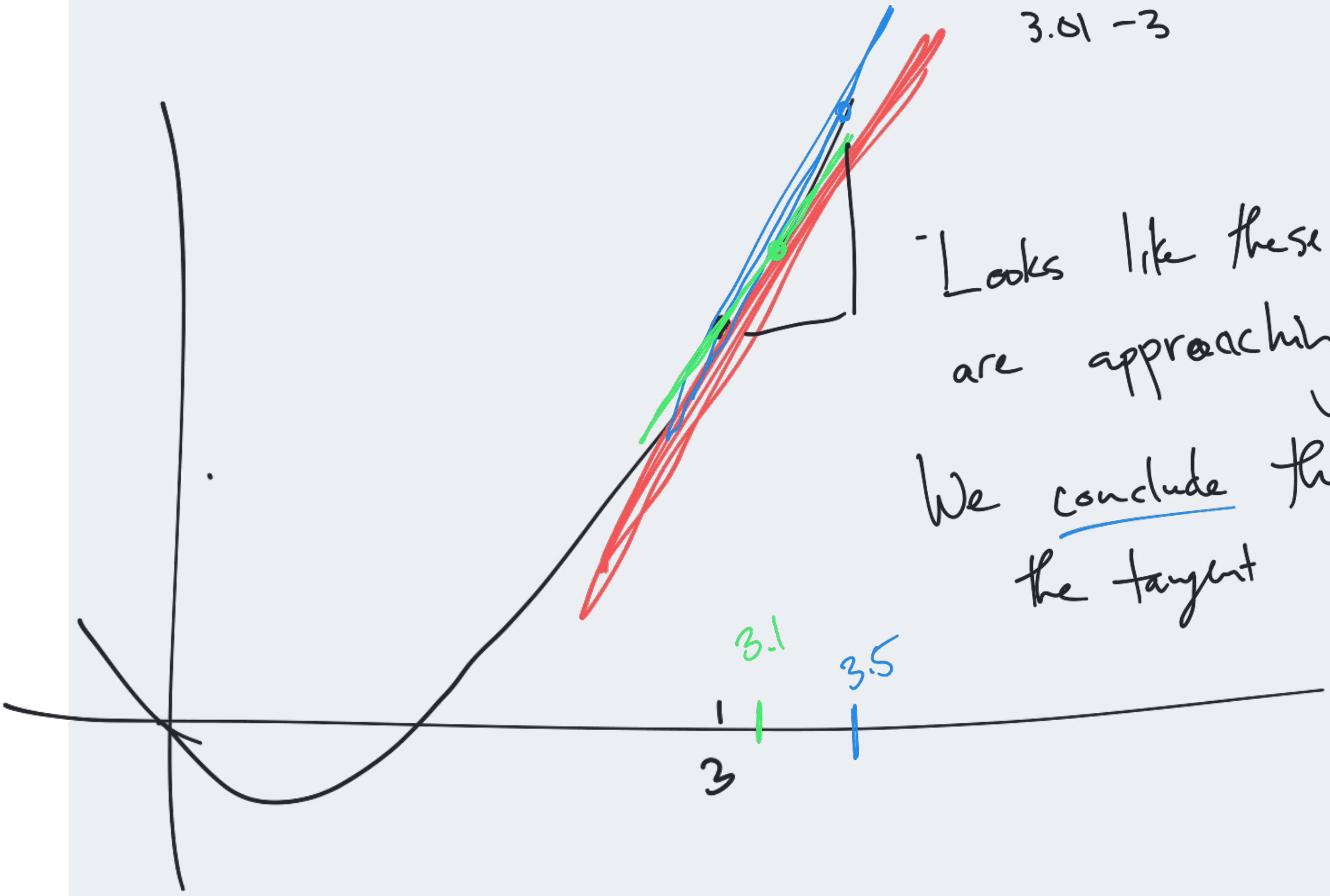
Let's measure the average slope between 2 points near $x=3$

3 & 3.5 $\leftarrow$ $\frac{\text{rise}}{\text{run}} = \frac{f(3.5)-f(3)}{3.5-3} = \frac{2.75}{.5} = 5.5$

or 3 & 3.1 $\leftarrow$ $\frac{f(3.1)-f(3)}{3.1-3} = \frac{.51}{.1} = 5.1$

or 3 & 3.01, etc.

$\frac{f(3.01)-f(3)}{3.01-3} = \frac{.0501}{.01} = 5.01$



Looks like these answers are approaching 5.

We conclude the slope of the tangent is 5.

This requires the concept of a

Limit

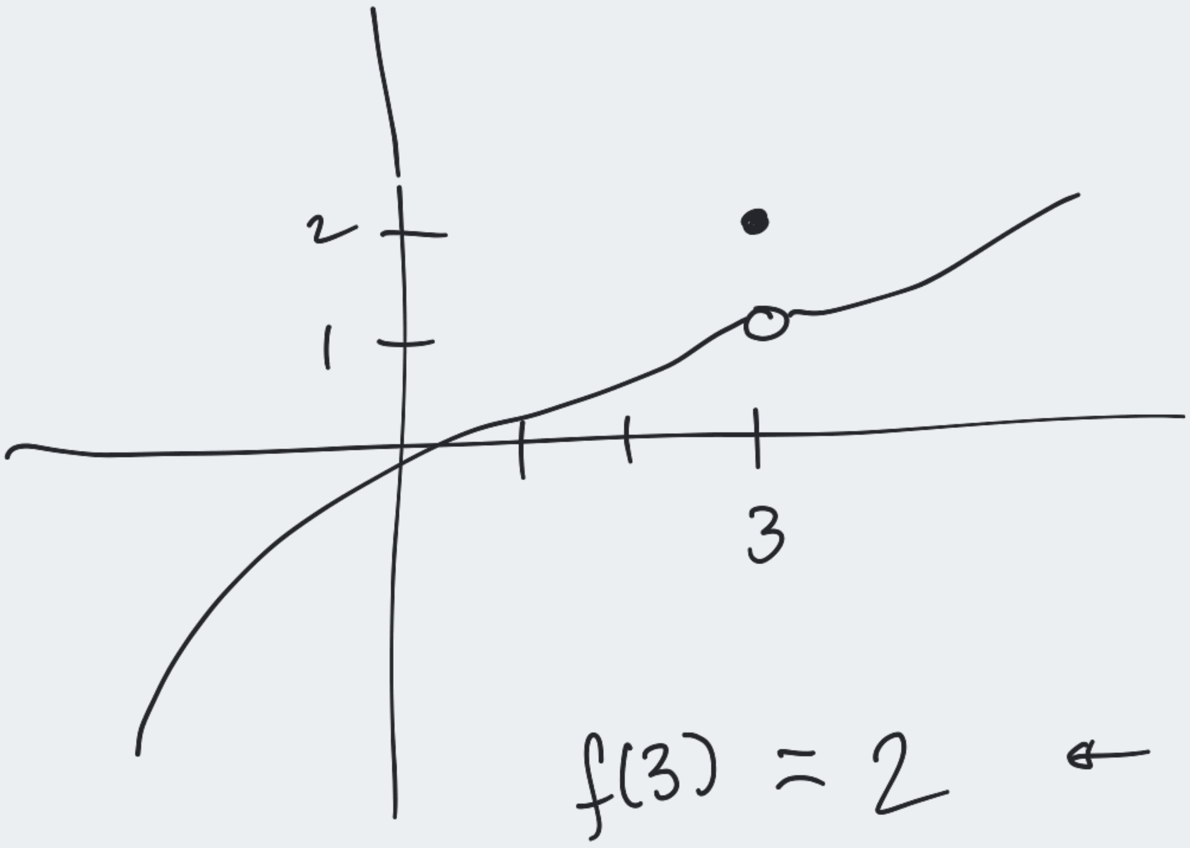
$\boxed{f(3)}$ means the y-value when $x=3$.

a limit is like: If x is close, but not equal to 3, then what is the y-value close to?

written as $\boxed{\lim_{x\to 3} f(x)}$

$f(3)$ means the y-val when $x=3$

$\lim_{x\to 3} f(x)$ means the y-val approached on $f(x)$ when x is near (but not equal) 3.

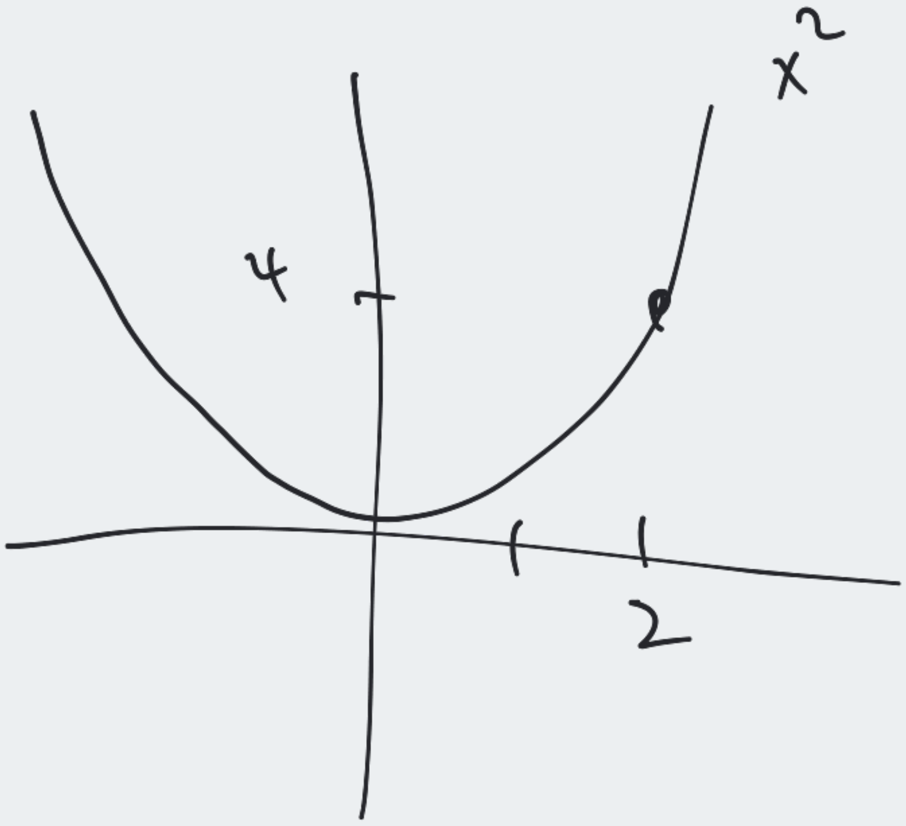


$f(3) = 2$ ← at the exact point

$\boxed{\lim_{x \to 3} f(x) = 1}$ ← along the curve, near the point.

↑ when x is really really close to 3, then y is really really close to 1

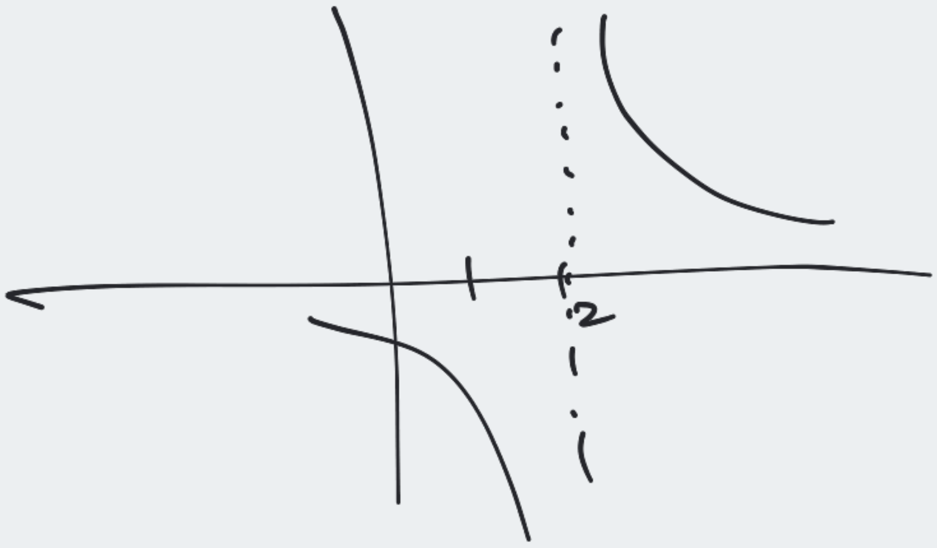
Usually the value at the point is the same as the value along the curve



$f(2) = 4$

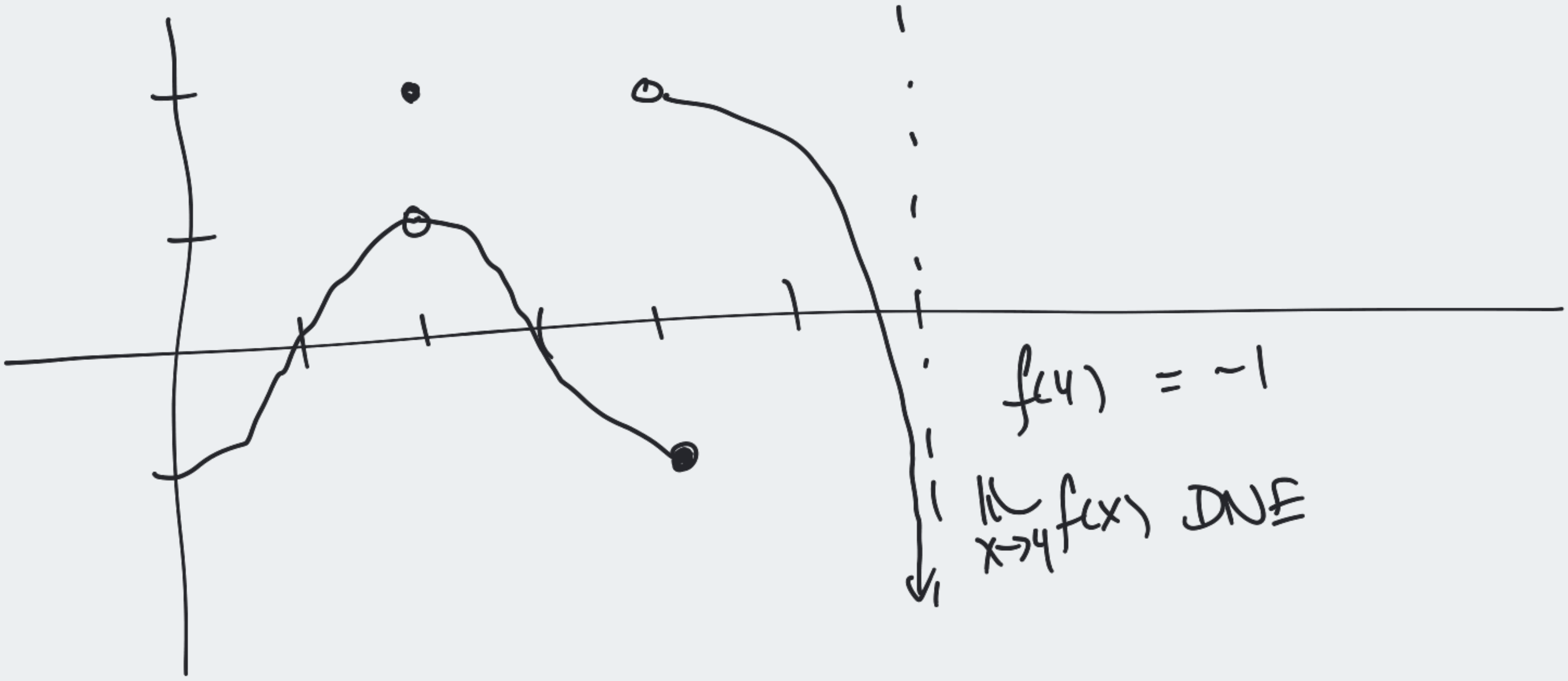
$\lim_{x \to 2} f(x) = 4$

$f(x) = \frac{1}{x-2}$



what is $\lim_{x \to 2} f(x)$? "does not exist" DNE

(curve does not approach a specific y-value)



$f(4) = -1$

$\lim_{x \to 4} f(x)$ DNE

Find

$f(2) = 2$ $\quad$ $f(3) = 0$ $\quad$ $f(5) = 1.6$ $\quad$ $f(6)$ DNE

$\lim_{x \to 2} f(x) = 1$ $\quad$ $\lim_{x \to 3} f(x) = 0$ $\quad$ $\lim_{x \to 5} f(x) = 1.6$ $\quad$ $\lim_{x \to 6} f(x)$ DNE

With formulas:

$$\lim_{x \to 1} \frac{x^2-1}{x-1}$$ we can try this using a calculator

try x-values close to 1.

try $x = 1.5 \to \frac{1.5^2-1}{1.5-1} = 2.5$

$x = 1.01$

$\hookrightarrow \frac{1.01^2-1}{1.01-1} = 2.01$

etc.

looks like $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

note: can't just use $x=1$:

$$\frac{1^2-1}{1-1} = \frac{0}{0}$$ is nothing

Could also graph it:

$$\lim_{x \to 0} \frac{\sin x}{x}$$

can't plug $x=0$: $\frac{\sin 0}{0} = \frac{0}{0}$

graph it, & look at values near $x=0$

I use Desmos.

Looks like when x is near 0, y is near 1.

Seems like $\lim_{x \to 0} \frac{\sin x}{x} = 1$.