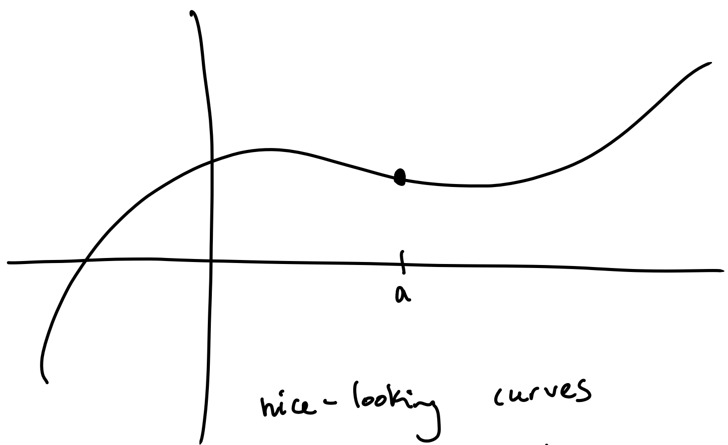
Limit Laws



nice-looking curves always have $\lim_{x \to a} f(x) = f(a)$

(always true for polynomials & rational funcs)

For polynomials & rational funcs, we find the limit by direct substitution

$$\lim_{x \to 1} 3x^2 - 7x + 1 = 3 \cdot 1^2 - 7 \cdot 1 + 1 = -3$$

For rational functions, $\frac{p(x)}{q(x)}$ you may plug in & get 0 in the denominator

if you get $\frac{c}{0}$ this means that $\lim_{x \to a} f(x)$ DNE.

except if you get $\frac{0}{0}$, we must use some trick to simplify.

First trick: factor & expand and cancel.

Ex1 $\lim_{x\to1} \frac{x^2-1}{x-1} \to \frac{1^2-1}{1-1} = \frac{0}{0}$ go back & simplify

$$= \lim_{x\to1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to1} x+1 = 1+1 = 2$$

Try: $\lim_{x\to1} \frac{x-1}{x^2-4x-5} = \frac{1-1}{1-4-5} = \frac{0}{-8} = 0$

$$\lim_{x\to3} \frac{x+4}{2x^2-x+3} = \frac{3+4}{2\cdot3^2-3+3} = \frac{7}{18}$$

$$\lim_{x\to1} \frac{x-1}{x^2+4x-5} \to \frac{0}{0}$$

$$\lim_{x\to1} \frac{\cancel{x-1}}{\cancel{(x-1)}(x+5)} = \lim_{x\to1} \frac{1}{x+5} = \frac{1}{1+5} = \frac{1}{6}$$

$$\cancel{\lim_{x\to1} \frac{1}{6}} \qquad \cancel{\lim_{x\to1} = \frac{1}{6}}$$

$$\lim_{x \to -1} \frac{(x+2)^2 - 1}{x+1} = \lim_{x \to -1} \frac{(x+2)(x+2) - 1}{x+1}$$

$$= \lim_{x \to -1} \frac{x^2 + 4x + 4 - 1}{x+1}$$

$$= \lim_{x \to -1} \frac{x^2 + 4x + 3}{x+1} = \lim_{x \to -1} \frac{(x+3)\cancel{(x+1)}}{\cancel{x+1}}$$

$$= \lim_{x \to -1} x + 3 = -1 + 3 = 2$$

For one-sided limits, do the same.

$$\lim_{x \to 2^-} x^2 + 4x = 2^2 + 4 \cdot 2 = 12$$

Interesting for piecewise funcs:

$$f(x) = \begin{cases} 3x+5 & \text{if } x \leq 3 \\ x^2 - 1 & \text{if } 3 < x \leq 5 \\ 24 & \text{if } x > 5 \end{cases}$$

$\lim_{x \to 3^-} f(x) = 3 \cdot 3 + 5 = 14$ (piece for x slightly less than 3)

$\lim_{x \to 3^+} f(x) = 3^2 - 1 = 8$ (. . . . . greater . . .)

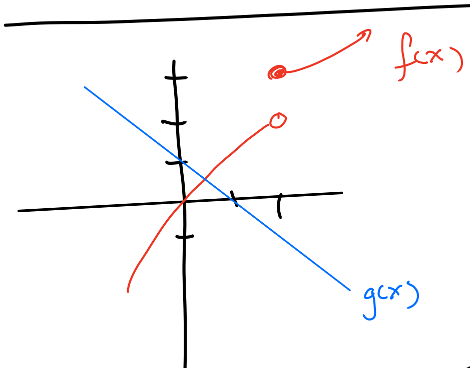
$\lim_{x \to 5^-} f(x) = 5^2 - 1 = 24$

$\lim_{x \to 5^+} f(x) = 24$ (3rd piece)

Limit laws: how lim plays with $+, -, \times, \ldots$

Theorem If $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$ exist, then

- $\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\lim_{x\to a}\left(f(x)-g(x)\right) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\lim_{x\to a}\left(c f(x)\right) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
- $\lim_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$



Find

$$\lim_{x\to 2^+}\left(f(x) - 3g(x)\right)$$

$$= \lim_{x\to 2^+} f(x) - \lim_{x\to 2^+} 3g(x)$$

$$= \lim_{x\to 2^+} f(x) - 3\lim_{x\to 2^+} g(x)$$

$$= 3 - 3\cdot(-1) = 6$$

One more limit law:

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

So "lim" works nice with roots.

This makes direct substitution work for any <u>algebraic</u> function (polyn with some roots)

<u>Ex!</u>

$$\lim_{x\to 2} \frac{\sqrt{x+5}}{3x} = \frac{\sqrt{2+5}}{3\cdot 2} = \frac{\sqrt{7}}{6}$$

A trick for these: "rationalize the radical"

If you see $\sqrt{a} - b$, multiply by $\frac{\sqrt{a}+b}{\sqrt{a}+b}$

$$\lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2} \xrightarrow{\text{plug in}} \frac{\sqrt{0^2+4} - 2}{0^2} = \frac{\sqrt{4}-2}{0} = \frac{0}{0}$$

go back & simplify

$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x^2\left(\sqrt{x^2+4}+2\right)}$$

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+4}\right)^2 - 2^2}{x^2\left(\sqrt{x^2+4}+2\right)}$$

$$= \lim_{x \to 0} \frac{x^2 \cancel{+4} \cancel{-4}}{x^2\left(\sqrt{x^2+4}+2\right)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+4}+2}$$

$$= \frac{1}{\sqrt{0+4}+2} = \frac{1}{2+2} = \frac{1}{4}$$

$$\frac{6}{\sqrt{3}} \qquad \frac{6\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} \qquad \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$