$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Ex/ The distance traveled by a falling object is:

$$s(t) = 4.9\,t^2 \quad , \quad t \text{ in sec}, \; s(t) \text{ in m}$$

a) If I drop a rock, how fast is it moving after 5 sec?

We'll find $s'(5)$

$$s'(5) = \lim_{h \to 0} \frac{s(5+h) - s(5)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(5+h)^2 - 4.9 \cdot 5^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9(25 + 10h + h^2) - 4.9 \cdot 25}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4.9 \cdot 25} + 49h + 4.9h^2 - \cancel{4.9 \cdot 25}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(49 + 4.9h)}{\cancel{h}} = \lim_{h \to 0} 49 + 4.9h$$

$$= 49 + 4.9 \cdot 0 = \boxed{49 \text{ m/s}}$$

b) If I drop it from a height of 3m, how fast is it going when it hits the ground?

Find the time of impact, then find derivative at that time.

$$s(t) = 4.9t^2$$

set dist $= 3$, solve for t.

$$4.9t^2 = 3$$

$$t^2 = 3/4.9$$

$$t = \sqrt{3/4.9} = .782$$

Now $s'(.782)$

$$s'(.782) = \lim_{h \to 0} \frac{s(.782+h) - s(.782)}{h}$$

$$= \lim_{h \to 0} \frac{4.9(.782+h)^2 - 4.9 \cdot .782^2}{h}$$

$$= \lim_{h \to 0} 4.9 \frac{\cancel{.782^2} + 1.56h + h^2 - \cancel{.782^2}}{h}$$

$$= \lim_{h \to 0} 4.9 \frac{\cancel{h}(1.56 + h)}{\cancel{h}}$$

$$= 4.9(1.56 + 0) = 7.66 \text{ m/s.}$$

Math Colloquium

5-6

Downstairs Library

Mathematical Billiards

The Derivative as a function

We can find the derivative without plugging in a specific point, we get a function as the answer.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(never plug in for x, so x's remain in the answer)

The derivative is like a function transformation:

start with a function $f(x)$,

obtain another function $f'(x)$.

How fast $f(x)$ is changing.

Ex| My bank account balance is given by:

$$f(t) = 2000 + 5/t \quad , \quad \begin{array}{l} t \text{ in days} \\ f(t) \text{ in } \$. \end{array}$$

Find $f'(t)$ ← how much my account changes on day t.

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h\to 0} \frac{2000 + \frac{5}{t+h} - \left(2000 + \frac{5}{t}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2000} + \frac{5}{t+h} - \cancel{2000} - \frac{5}{t}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{5}{t+h} - \frac{5}{t} \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \left(\frac{5}{(t+h)} \cdot \frac{t}{t} - \frac{5}{t} \cdot \frac{t+h}{t+h} \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \left(\frac{5t}{t(t+h)} - \frac{5t+5h}{t(t+h)} \right)$$

$$= \lim_{h\to 0} \frac{1}{h} \left(\frac{5t - (5t+5h)}{t(t+h)} \right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \cdot \frac{-5\cancel{h}}{t(t+h)}$$

$$= \lim_{h\to 0} \frac{-5}{t(t+h)} = \boxed{\frac{-5}{t^2}}$$

$f(t) = 2000 + 5/t$

$f'(t) = -5/t^2$

$f(1) = 2005$ ← on day 1, I have \$2005

$f'(1) = -5$ ← on day 1, I'm losing \$5/day.

Find $f'(x)$:

$f(x) = 5 + 2x - x^2$	$2 - 2x$
$f(x) = 3 - 7x$	-7

$$\lim_{h\to 0} \frac{5+2(x+h)-(x+h)^2 - (5+2x-x^2)}{h}$$

$$= \lim_{h\to 0} \frac{5+2x+2h-(x^2+2xh+h^2)-5-2x+x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{5}+\cancel{2x}+2h-\cancel{x^2}-2xh-h^2-\cancel{5}-\cancel{2x}+\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2h - 2xh - h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(2-2x-h)}{\cancel{h}}$$

$$= 2 - 2x - 0 = \underline{2-2x}$$

$f(x) = 3 - 7x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3 - 7(x+h) - (3-7x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3} - \cancel{7x} - 7h - \cancel{3} + \cancel{7x}}{h} = \lim_{h\to 0} \frac{-7h}{\cancel{h}} = \boxed{-7}$$

Harder $\qquad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

$$f(x) = \frac{x+3}{x-1}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h+3}{x+h-1} - \frac{x+3}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{x+h+3}{x+h-1} \cdot \frac{x-1}{x-1} - \frac{x+3}{x-1} \cdot \frac{x+h-1}{x+h-1} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{x^2 + hx + 3x + -x - h - 3 - (x^2 + hx - x + 3x + 3h - 3)}{(x+h-1)(x-1)} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{\cancel{x^2} + \cancel{hx} + \cancel{3x} - \cancel{x} - h - \cancel{3} - \cancel{x^2} - \cancel{hx} + \cancel{x} - \cancel{3x} - 3h + \cancel{3}}{(x+h-1)(x-1)} \right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \left(\frac{-4\cancel{h}}{(x+h-1)(x-1)} \right)$$

$$= \frac{-4}{(x-1)(x-1)} = \frac{-4}{(x-1)^2}$$

Deriv. is a limit of something, so it might not exist.

$$f(x) = \frac{x+3}{x-1}$$

$$f'(x) = \frac{4}{(x-1)^2}$$ ← this DNE if $x = 1$

We say f is <u>not differentiable</u> at $x=1$

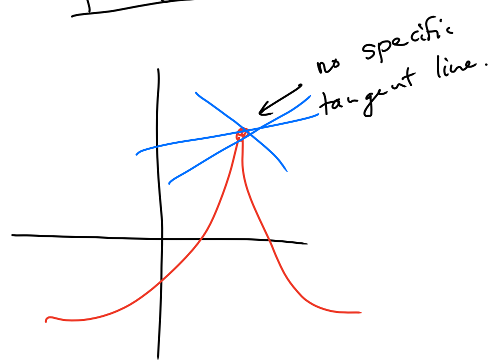
↑ derivative DNE

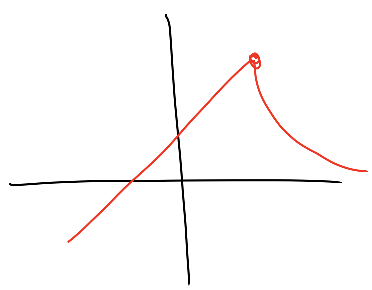
Here also $f(1)$ DNE

So when $f(a)$ DNE, then $f'(a)$ also DNE.

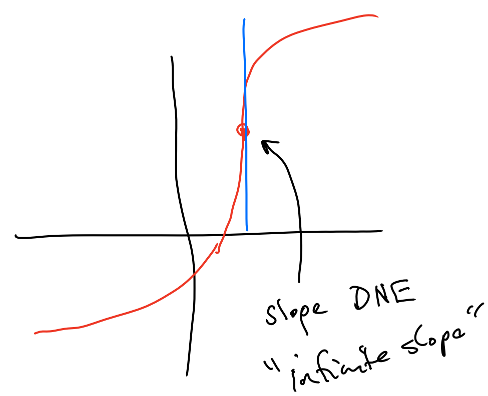
But sometimes f' DNE even when $f(a)$ does exist.

<u>pointy-point</u>





<u>vertical slope</u>



<u>MPG ZERO</u>

$$\frac{\text{miles}}{\text{gal}} = \frac{\quad}{0}$$