Derivatives in applied problems

In physics: if $s(t)$ is position,

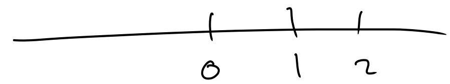
$s'(t) = v(t)$ is velocity

$s''(t) = v'(t) = a(t)$ is acceleration

Say we have a particle, moving according to:

$$s(t) = t^3 - 6t^2 + 9t$$

moving like:



Lots of q's:

- A function for velocity:

$$v(t) = 3t^2 - 12t + 9$$

- How fast & in what direction is it moving at $t = 2$ sec.

$$v(2) = 3 \cdot 2^2 - 12 \cdot 2 + 9 = -3 \text{ m/s}$$

3 m/s, moving left.

- Is it ever at rest?

set $v(t) = 0$, solve for t.

$$v(t) = 3t^2 - 12t + 9$$

$$3t^2 - 12t + 9 = 0$$

$$t^2 - 4t + 3 = 0$$

$$(t-3)(t-1) = 0$$

$$t = 3 \quad \& \quad t = 1.$$

it is at rest at time 1 sec & 3 sec.

- At which times is it moving right? (or left?)

means $v(t)$ is positive.

when is $v(t) > 0$?

when is $3(t-3)(t-1) > 0$

$$(t-3)(t-1) > 0$$

So $t-3$ & $t-1$ are either both pos or both neg.

Both pos: $t-3 > 0$ and $t-1 > 0$

$t > 3$ and $\cancel{t > 1}$ redundant

Both neg: $t-3 < 0$ and $t-1 < 0$

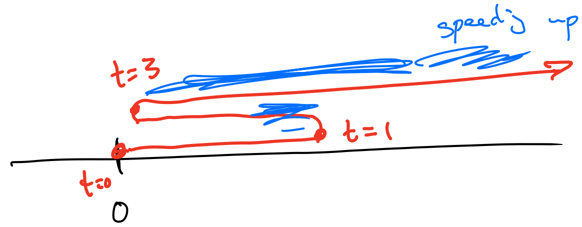
$\cancel{t < 3}$ and $t < 1$

So $v(t) > 0$ when $t > 3$, or $t < 1$.

$v(t) < 0$ otherwise: when $1 < t < 3$.

moving right for t in $(-\infty, 1)$ & $(3, \infty)$

left for t in $(1, 3)$.



- At which times is it speeding up vs slowing down?

When is acceleration direction the same as velocity direction? $\Big\}$ speeding up.

Could have v & a both pos,

or v & a both neg.

Let's say when $a(t)$ is pos vs neg:

$$v(t) = 3t^2 - 12t + 9$$

$$a(t) = 6t - 12$$

$a(t) > 0$ when: $6t - 12 > 0$

$$6t > 12$$

$$t > 2$$

a is pos when $t > 2$

neg when $t < 2$.

Speeding up: a & v both pos or both neg:

a(t) & v(t) both pos:

a is pos when $t > 2$

v is pos when $t > 3$ or $t < 1$

So a & v both pos when:

$t > 2$ and ($t > 3$ or $t < 1$)

i.e. $t > 3$. t is in $(3, \infty)$.

a(t) & v(t) both neg:

a is neg when $t < 2$

v is neg when $1 < t < 3$.

both neg when $t < 2$ and $1 < t < 3$.

i.e: $1 < t < 2$ t is in $(1, 2)$

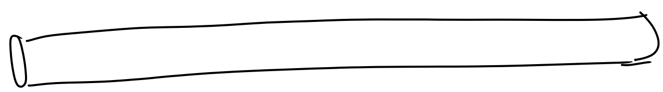
Speeding up for t in $(1,2)$ & $(3, \infty)$.

Slowing down for t in $(0,1)$ & $(2,3)$

2 more examples using derivative

without time?

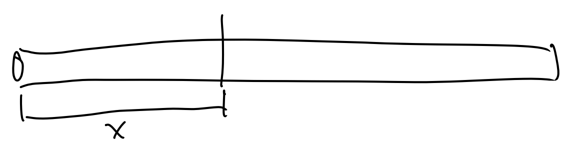
A nonhomogeneous rod



the mass is not evenly distributed.

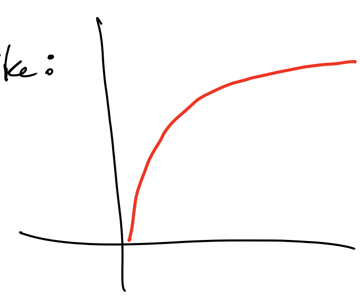
To describe the mass, we have a function

$m(x)$ = total mass up to position x.

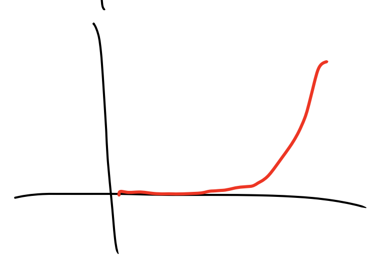


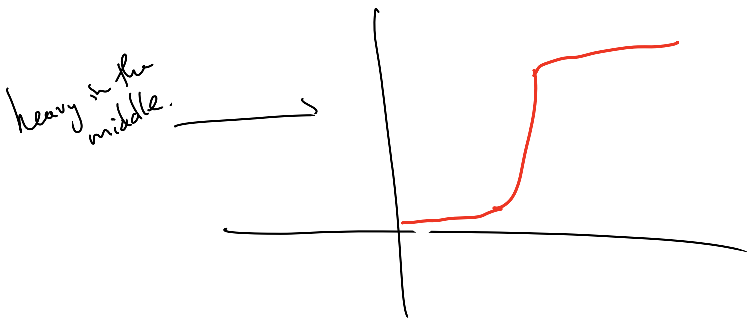
$m(x)$ is the mass of this piece.

"heavier on the left" would look like:



heavier on the right

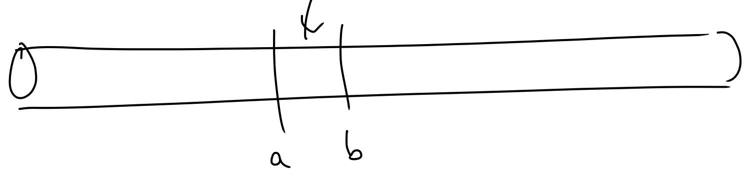




What is the meaning of $m'(x)$?

$m'(x)$ is like

$$\frac{m(b)-m(a)}{b-a} = \frac{\text{mass of the piece}}{\text{width of the piece}}$$



Here, $m'(x)$ is the (linear) density of the rod at various points.

Economics!

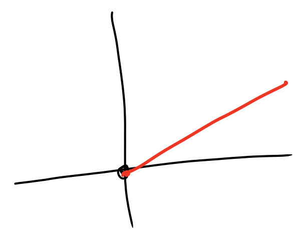
Say I'm making jars of maple syrup

Say my costs are:

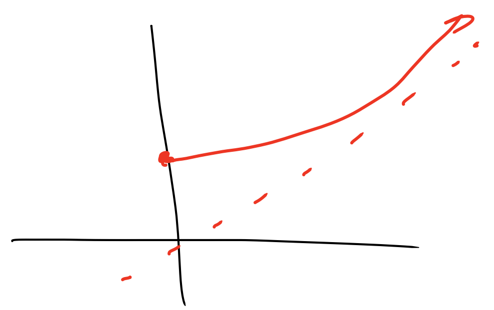
$$C(x) = \text{cost to make } x \text{ jars.}$$

very simple: it costs 75¢ per jar:

$$C(x) = .75x$$



More realistically it might look like



What does $C'(x)$ represent?

$C'(x)$ is like:

$$\frac{C(x+h) - C(x)}{h} = \frac{(\text{cost of } x+h) - (\text{cost of } x)}{h}$$

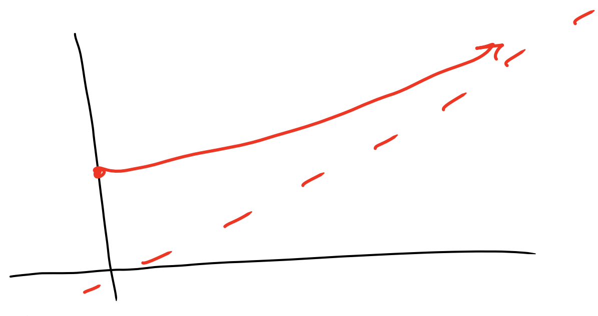
$$= \frac{\text{additional cost to make } h \text{ more jars}}{h}$$

$C'(x)$ represents: the extra cost per jar to make additional jars, when we already made x.

This is the <u>marginal cost</u>

<u>Ex1</u> My jars cost:

$$C(x) = 2 + \frac{3x^2}{4x+100}$$



marginal cost:

$$C'(x) = \frac{(4x+100)\cdot 6x - 3x^2 \cdot 4}{(4x+100)^2}$$

$$= \frac{24x^2 + 600x - 12x^2}{(4x+100)^2} = \frac{12x^2+600x}{(4x+100)^2}$$

Find marg. cost at $x = 1, 10, 100$.

$$C'(1) = \frac{12+600}{(4+100)^2} = .056$$

$$C'(10) = \frac{12\cdot 10^2 + 600\cdot 10}{(4\cdot 10+100)^2} = .36$$

$$C'(100) = \qquad = .72$$