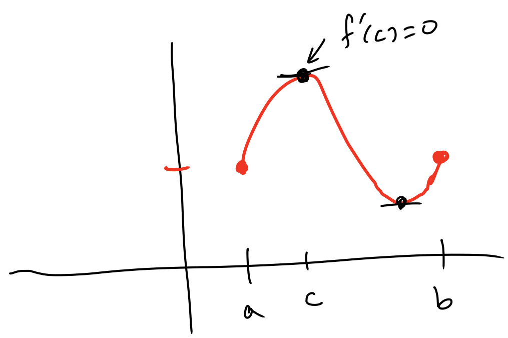
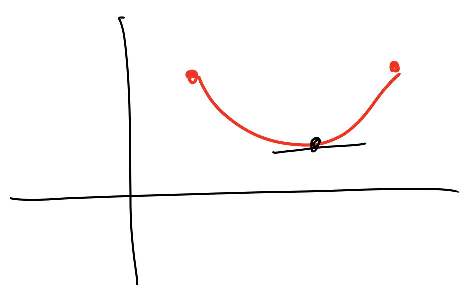
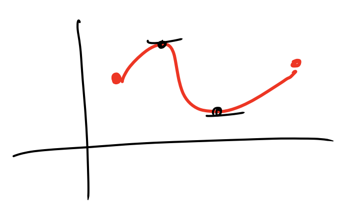
Mean Value Theorem

Based on a simpler idea: Rolle's Thm

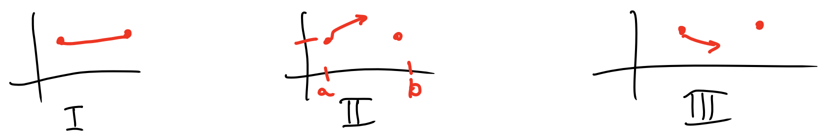
Rolle's Theorem If f is differentiable on $[a,b]$, and $f(a)=f(b)$, then there is some c in (a,b) with $f'(c)=0$.







Proof: 3 cases, according to which way the curve starts out:



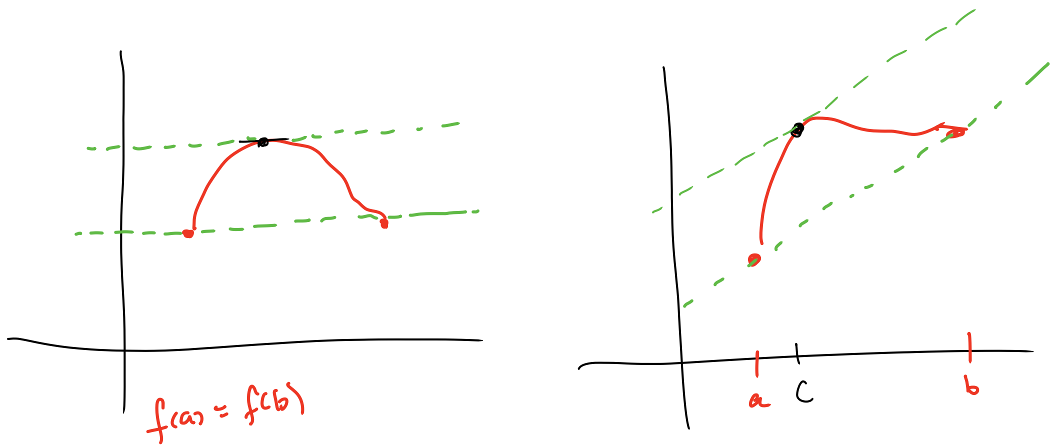
Case I: here, the curve $f(x)$ is always horizontal, so in fact all points have slope 0. Shown!

Case II: We go up from $f(a)$, so $f(a)$ is not the abs maximum, also $f(b) = f(a)$, so $f(b)$ is not the abs max. So the abs max does not occur at the endpts, so it must be at a critical #, so it's at some point c where $f'(c) = 0$. Shown!

Case III: Same, but with "min" instead of "max".

Mean Value Theorem

a fancier version of Rolle's Thm:



there is some c in (a,b)

where $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

avg. slope from a to b

Mean Value Theorem

Thm If f is differentiable on $[a,b]$,

then there is some c in (a,b)

with

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

This is generalization of Rolle's Theorem.

Ex1 $f(x) = x^3 - 2x^2 + 3$, on $[-1, 2]$

what does MVT say?

MVT says there is some c in $(-1, 2)$

with $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{3 - 0}{2 - -1} = \frac{3}{3} = 1$,

$f(b) = f(2) = 2^3 - 2 \cdot 2^2 + 3 = 3$

$f(a) = f(-1) = (-1)^3 - 2(-1)^2 + 3$

$= -1 - 2 + 3 = 0$

So there's some point in $(-1, 2)$

where $f'(x) = 1$.

The speed limit is 65 mph,

I pass a toll booth at 9:07 AM,

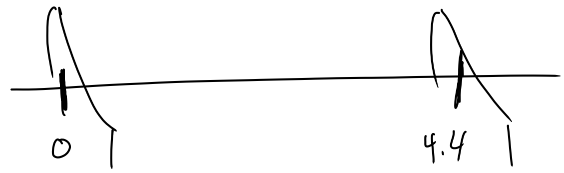
then another toll 4.4 miles later at 9:11 AM.

Prove that I exceeded the speed limit at some point,

Let $f(t)$ be distance traveled at t minutes after 9AM.

9:07 AM: $f(7) = 0$

$f(11) = 4.4$



use MVT on $[7, 11]$

there is some point c where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{4.4 - 0}{11 - 7} = \frac{4.4}{4} = 1.1$$

So at some time, my speed was 1.1 miles/min

= 66 miles/hr.

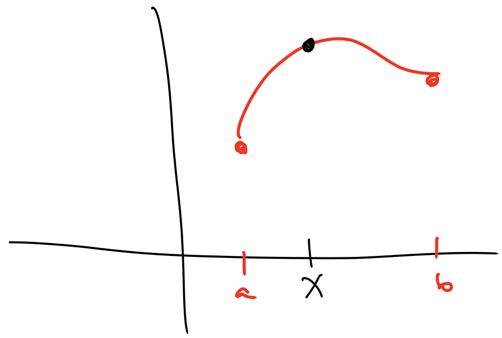
Busted!

Fun fact based on MVT:

Thm If $f'(x) = 0$ always,

then $f(x)$ is a constant function.

Pf/ Consider MVT on $[a, x]$:



$$f'(c) = \frac{f(x) - f(a)}{x - a}$$

but $f'(c)$ must be 0,

so
$$0 = \frac{f(x) - f(a)}{x - a}$$

so $\quad 0 = f(x) - f(a)$

so $\quad f(x) = f(a)$ for every x in $[a, b]$.

So all values are equal to $f(a)$.

so f is a constant!

Corollary If $f'(x) = g'(x)$, then
$f(x) = g(x) + C$ for a constant.

Pf If $f'(x) = g'(x)$, then

$$f'(x) - g'(x) = 0$$

so $\quad f(x) - g(x)$ is a constant

So $f(x) - g(x) = C$

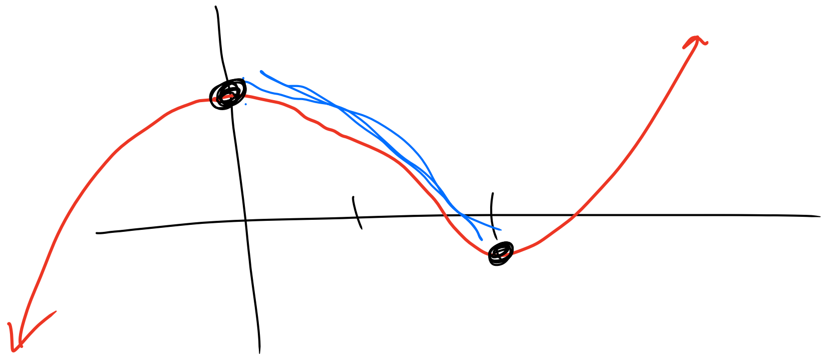
So $f(x) = g(x) + C$ Shown.

Derivatives & the shape of the graph

Most basic: increasing vs decreasing.

Increasing means $f'(x) > 0$

Decreasing means $f'(x) < 0$.



On which intervals (x-values) is it inc or dec?

It's decreasing on $(0,2)$

increasing on $(2,\infty)$ & $(-\infty,0)$