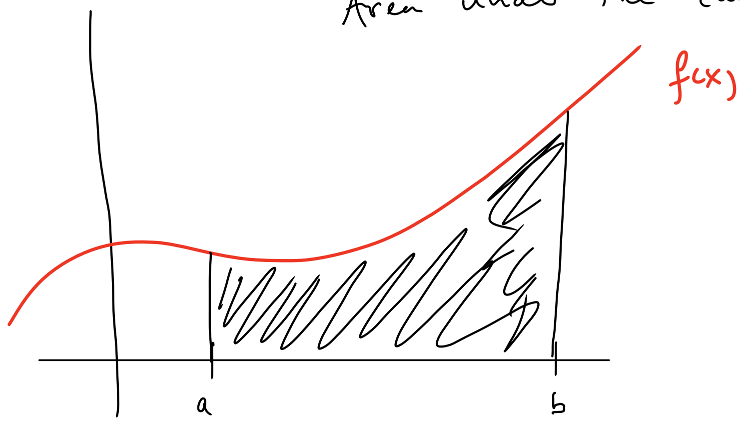
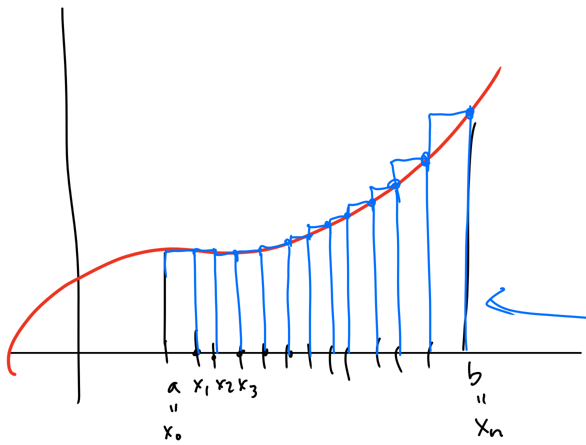
# Area under the curve




chop the interval into $n$ bits.

"rectangles on the right endpts"

interval from a to b

The sum of these rectangle areas is $R_n$

left side would be $L_n$

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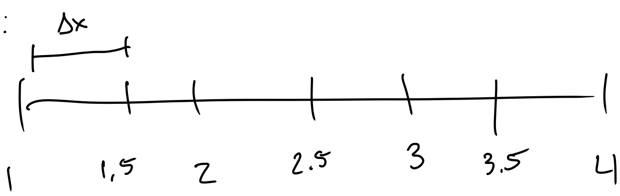
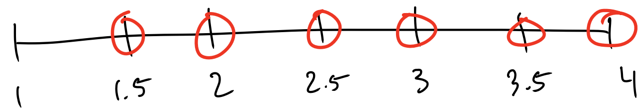
<u>Ex|</u> Find $R_6$ & $L_6$ for $f(x) = x^2 - 2x$ on $[1, 4]$

the interval:



to find the points: $\Delta x = \frac{b-a}{n} = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

For $R_6$:




Add up base · height

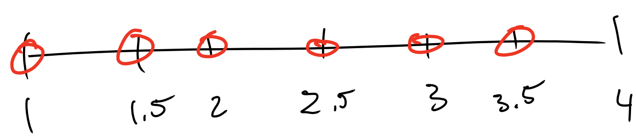
$\uparrow$ $\Delta x = \frac{1}{2}$    $\uparrow$ $f(x_i)$

So $R_6 = \frac{1}{2} \cdot f(1.5) + \frac{1}{2} \cdot f(2) + \frac{1}{2} f(2.5) + \frac{1}{2} \cdot f(3) + \frac{1}{2} \cdot f(3.5) + \frac{1}{2} f(4)$

$= \frac{1}{2}(1.5^2 - 2 \cdot 1.5) + \frac{1}{2}(2^2 - 2 \cdot 2) + \dots + \frac{1}{2}(4^2 - 2 \cdot 4)$

$= \underline{4.1875}$

$L_6$:




$L_6 = \frac{1}{2} f(1) + \frac{1}{2} f(1.5) + \dots + \frac{1}{2} f(3.5)$

$= \frac{1}{2}(1^2 + 2 \cdot 1) + \frac{1}{2}(1.5^2 + 2 \cdot 1.5) + \dots + \frac{1}{2}(3.5^2 + 2 \cdot 3.5)$

$= \underline{1.9375}$

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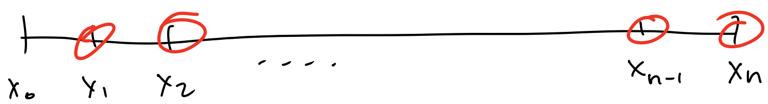
For more accuracy, increase the n!

So, the true area will be

$\boxed{\lim_{n \to \infty} R_n}$ (if the limit exists)

Set it up more abstractly




Then $R_n = \Delta x f(x_1) + \Delta x f(x_2) + \dots + \Delta x f(x_n)$

$$= \sum_{i=1}^{n} \Delta x f(x_i)$$

$$\boxed{R_n = \sum_{i=1}^{n} f(x_i)\Delta x} \qquad \boxed{L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x}$$

We get true area by doing $\lim_{n\to\infty} R_n$ or $\lim_{n\to\infty} L_n$,

this is the definite integral.

Def The definite integral of $f(x)$ on $[a,b]$ is

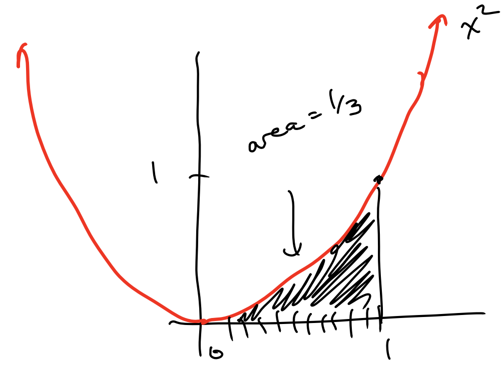
written like $\int_a^b f(x)dx$, defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\Delta x}_{\text{a "Riemann sum"}}$$

where $\Delta x = \frac{b-a}{n}$, and $x_i$ are the endpoints of the intervals when we divide $[a,b]$ into $n$ pieces.

In simple examples, we can actually compute this limit.

Lets do $\int_0^1 x^2 dx$




$$\int_0^1 x^2 dx = \lim_{n\to\infty} R_n$$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\boxed{x_i = x_0 + i\,\Delta x}$$

$$x_i = 0 + i \cdot \frac{1}{n}$$

$$x_i = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3}$$

in a sum, we can factor any constant out

no i's

$$= \lim_{n\to\infty} \frac{1}{n^3} \boxed{\sum_{i=1}^{n} i^2}$$

?

Tricks for sums:

$\sum_{i=1}^{n} i \qquad \sum_{i=1}^{n} i^2 \qquad \sum_{i=1}^{n} i^3 \qquad \sum_{i=1}^{n} 1$

easy: $\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\dots+1}_{n \text{ times}} = n$

med: $\sum_{i=1}^{n} i = 1+2+3+4+\dots+n$

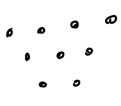
cute trick: to do 1 to 50:

$1+2+3+\dots+48+49+50$

each group adds to 51 $(n+1)$

# of groups is 25 $(n/2)$

The sum is $\frac{n}{2}\cdot(n+1) = \boxed{\frac{n(n+1)}{2}}$



$1+2+3+4 = 10$

$$\boxed{\begin{array}{ll} \sum_{i=1}^{n} 1 = n & \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} & \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \end{array}}$$

Finish $\displaystyle\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n} i^2$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n\to\infty} \frac{2n^3 + \dots}{6n^3}$$

$$= \frac{2}{6} = \frac{1}{3}$$

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Let's try $\displaystyle\int_3^5 x^2\,dx$

$$\boxed{\begin{aligned} \Delta x &= \frac{b-a}{n} \\ x_i &= a + i\Delta x \end{aligned}}$$

$$\int_3^5 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\Delta x = \frac{5-3}{n} = \frac{2}{n}$$

$$x_i = 3 + i\cdot\frac{2}{n} = 3 + \frac{2i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(3+\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$\left(3+\frac{2i}{n}\right)\left(3+\frac{2i}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{2i}{n}\right)^2\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(9 + \frac{12i}{n} + \frac{4i^2}{n^2}\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{18}{n} + \frac{24i}{n^2} + \frac{8i^2}{n^3}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{18}{n} + \sum_{i=1}^{n} \frac{24i}{n^2} + \sum_{i=1}^{n} \frac{8i^2}{n^3}$$ break up the sum

$$= \lim_{n\to\infty} \frac{18}{n}\sum_{i=1}^{n} 1 + \frac{24}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{18}{n}\cdot n + \frac{24}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} 18 + 12\,\frac{n(n+1)}{n^2} + 8\cdot\frac{n(n+1)(2n+1)}{6n^3}$$

$$= 18 + 12\cdot\frac{1}{1} + 8\cdot\frac{2}{6}$$

$$= \boxed{30 + 8/3}$$