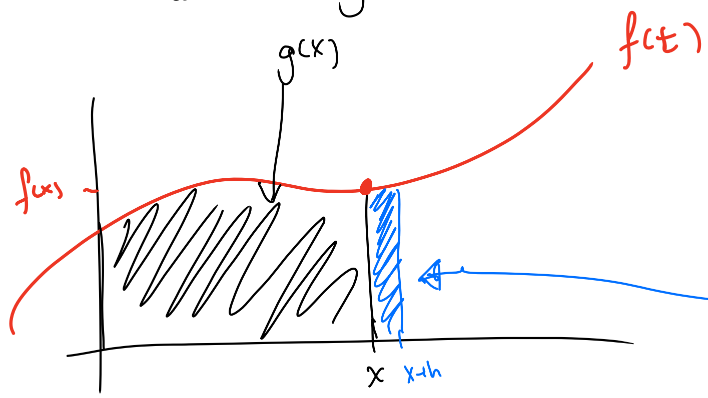
Fundamental Theorem of Calculus

if $g(x) = \int_a^x f(t)\,dt$,

what is $g'(x)$?



$g'(x)$ measures this

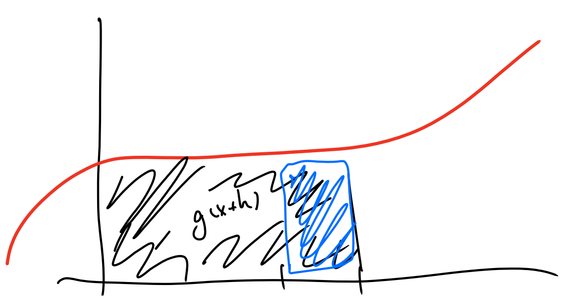
Theorem (FTC #1) If f is continuous,

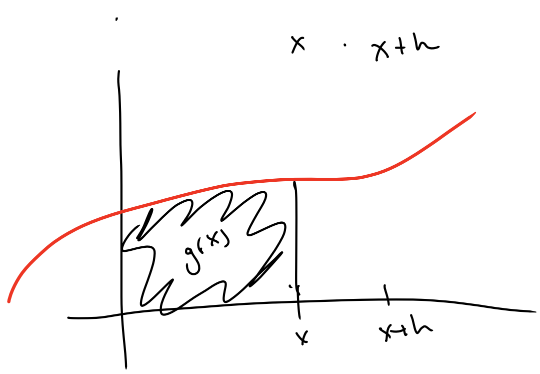
then $\frac{d}{dx}\underbrace{\int_a^x f(t)\,dt} = f(x)$

Pf let $g(x) = \int_a^x f(t)\,dt$,

we want to show $g'(x) = f(x)$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(g(x+h) - g(x)\right)$$





$g(x+h) - g(x)$ is the skinny rectangle:

so $g'(x) = \lim_{h\to 0} \frac{1}{h}\left(\text{area of blue rectangle}\right)$

$= \lim_{h\to 0} \frac{1}{h}\left(h \cdot f(x)\right) = f(x)$

Shown.

FTC #1: $\boxed{\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)}$

Ex1 if $g(x) = \int_2^x t^2+5t\,dt$, find $g'(3)$.

Find $g'(x)$: $g'(x) = x^2+5x$

So $g'(3) = 3^2+5\cdot 3 = 24$.

$\frac{d}{dx}\sin x = \cos x$

Ex1 $\qquad g(x) = \int_0^{x^2} 5t^4\,dt$, find $g'(4)$.

use chain rule: "inside" is x^2
outside is the integral

$$g'(x) = \left(\begin{array}{c}\text{deriv of outside} \\ \text{with inside plugged in}\end{array}\right) \cdot \left(\text{deriv of inside}\right)$$

$$= 5(x^2)^4 \cdot 2x$$

$$= 5 \cdot x^8 \cdot 2x = 10x^9$$

so $g'(4) = 10 \cdot 4^9$

FTC #2 (the better one)

it's about $\int_a^b f'(x)\,dx$

let $g(x) = \int_a^x f'(t)\,dt$, then

we know by FTC #1,

$g'(x) = f'(x)$ so g & f have same deriv, so they differ by a constant.

So $g(x) = f(x) + C$

$g(x) = \int_a^x f'(t)\,dt$

we can determine the C:

$$g(a) = \int_a^a f'(t)\,dt = 0$$

plug $x=a$, set $=0$

$$0 = f(a) + C \qquad \text{so} \qquad C = -f(a).$$

So $g(x) = f(x) - f(a)$, plug $x=b$

$$g(b) = f(b) - f(a)$$

so $$\int_a^b f'(t)\,dt = f(b) - f(a)$$

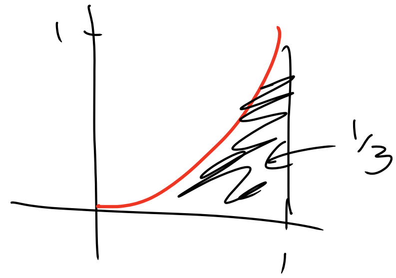
Thm FTC #2 If F is the antideriv of f,

then $$\boxed{\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)}_{\text{antiderivs. Easy! (if you know how)}}}$$

def. integral HARD!

$$\int_0^1 x^2\,dx = F(1) - F(0)$$

where $F(x) = \frac{1}{3}x^3 + C$



$$= \frac{1}{3}\cdot 1^3 + C - \left(\frac{1}{3}\cdot 0^3 + C\right)$$

$$= \frac{1}{3} + C - 0 - C = \boxed{\frac{1}{3}}$$

Special way to write it:

$$\int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big]_0^1$$

do antideriv, no +C,

$\Big]_0^1$ means will plug in & subtract

$$= \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}$$

$$\int_2^4 3x-2\,dx = 3\cdot\frac{1}{2}x^2 - 2x\Big]_2^4$$

$$= 3\cdot\frac{1}{2}\cdot 4^2 - 2\cdot 4 - \left(3\cdot\frac{1}{2}\cdot 2^2 - 2\cdot 2\right)$$

$$\int_1^3 2x^2 + \frac{1}{x^4}\,dx = \int_1^3 2x^2 + x^{-4}\,dx = 2\cdot\frac{1}{3}x^3 + \frac{1}{-3}x^{-3}\Big]_1^3$$

$$= 2\cdot\frac{1}{3}\cdot 3^3 + \frac{1}{-3}\cdot 3^{-3} - \left(2\cdot\frac{1}{3}\cdot 1^3 + \frac{1}{-3}\cdot 1^{-3}\right)$$

$$\int_{-1}^1 (2x+3)^2\,dx = \int_{-1}^1 (2x+3)(2x+3)\,dx$$

$$= \int^1 4x^2 + 12x + 9\,dx$$

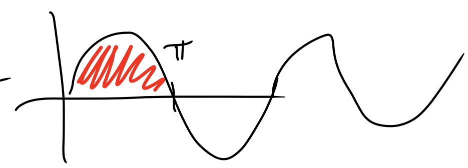
$\int_{-1}$

$$= 4\cdot\frac{1}{3}x^3 + 12\cdot\frac{1}{2}x^2 + 9x \Big]_{-1}^{1}$$

$$= 4/3\cdot 1^3 + 6\cdot 1^2 + 9\cdot 1 - \left(\frac{4}{3}\cdot(-1)^3 + 6(-1)^2 + 9(-1)\right)$$

$$\int_5^{10} \frac{x^2 + 7\sqrt{x}}{x^5}\,dx = \int_5^{10} x^{-3} + 7x^{-4.5}\,dx$$

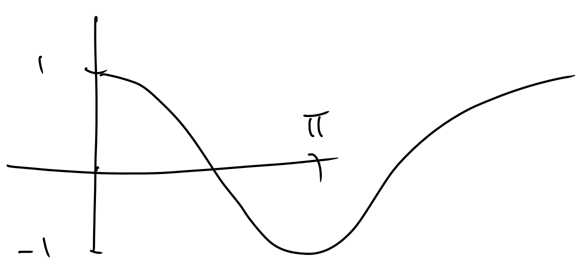
$$= \frac{1}{-2}x^{-2} + 7\cdot\frac{1}{-3.5}x^{-3.5} \Big]_5^{10} =$$



$$\int_0^{\pi} \sin x\,dx$$

$$= -\cos x \Big]_0^{\pi} = -\cos\pi - (-\cos 0)$$

$$= -(-1) - (-1) = 2$$



Properties of the definite integral

- $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

  Super Important

- $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

- $\int_a^a f(x)\,dx = 0$

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

- if $f(x) \geq g(x)$ on $[a, b]$,

  then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$