Math 3371

Homework #9

#1a x^3 is a polynomial, so it's continuous.

1b Let $x_n = n + 1/n$
$y_n = n.$

Then $|x_n - y_n| = 1/n \to 0$

but $|f(x_n) - f(y_n)| = |(n+1/n)^3 - n^3|$

$= |\cancel{n^3} + 3n + 3\cdot\frac{1}{n} + \frac{1}{n^3} - \cancel{n^3}|$ diverges (because of the $3n$)

So $|f(x_n) - f(y_n)| \not\to 0.$

1c Say A is bounded above (in abs. value) by M.

Let $\varepsilon > 0$, will find $\delta > 0$ s.t. $\forall x, y \in A$ we have
$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$$|f(x) - f(y)| = |x^3 - y^3| = |x-y|\,|x^2 + xy + y^2|$$

$$\leq |x-y| \cdot (M^2 + M^2 + M^2)$$

$$\leq |x-y| \cdot 3M^2$$

Let $\delta = \varepsilon / 3M^2$. Then if $0 < |x-y| < \delta$, we have

$|f(x) - f(y)| \leq |x-y| \cdot 3M^2 < \frac{\varepsilon}{3M^2} \cdot 3M^2 = \varepsilon$ as desired.

#3 $f(x) = \frac{1}{x^2}$ is unif. cont. on $[1,\infty)$:

Let $\varepsilon > 0$ be given, will find $\delta > 0$ s.t. $\forall x,y \in [1,\infty)$:

$$0 < |x-y| < \delta \Rightarrow \left|\frac{1}{x^2} - \frac{1}{y^2}\right| < \varepsilon.$$

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2-x^2}{x^2y^2}\right| = |x-y| \cdot \left|\frac{x+y}{x^2y^2}\right|$$

$$= |x-y| \cdot \left|\frac{x}{x^2y^2} + \frac{y}{x^2y^2}\right|$$

$$= |x-y| \cdot \left|\frac{1}{xy^2} + \frac{1}{x^2y}\right| \le |x-y| \cdot \left|\frac{1}{1\cdot 1^2} + \frac{1}{1^2\cdot 1}\right|$$

$$\le |x-y| \cdot 2$$

Let $\delta = \varepsilon/2$. Then when $|x-y| < \delta$, we have

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \le |x-y| \cdot 2 < \varepsilon/2 \cdot 2 = \varepsilon$$ as desired.

#8a domain $[0,1]$, range $(0,1)$ is impossible

since $[0,1]$ is compact, the image set must be compact. But $(0,1)$ is not compact.

#8b domain $(0,1)$, range $[0,1]$

is possible $\longrightarrow$