Math 3371 HW #12

6.2 #1d, 2a, 3b

6.3 #1a

6.2 #1d $\quad f_n(x) = \dfrac{nx}{1+nx^2}$ $\quad$ looking at graphs, pointwise it approaches $f(x) = 1/x$

It is uniform on $(1,\infty)$

Proof let $\varepsilon > 0$ be given, will find $n \in \mathbb{N}$ s.t. $\forall x \in (1,\infty)$,

$$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| = \left| \frac{nx^2 - (1+nx^2)}{(1+nx^2)\cdot x} \right|$$

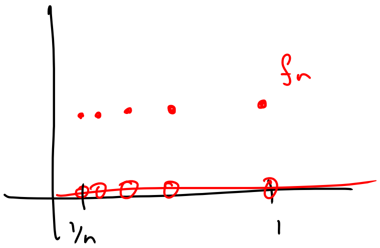
$$= \left| \frac{-1}{(1+nx^2)\cdot x} \right| < \frac{1}{|(1+n\cdot 1)\cdot 1|} = \frac{1}{1+n} < \frac{1}{n}$$

Let $N > 1/\varepsilon$. Then if $n > N$ we have:

$|f_n(x) - f(x)| < \frac{1}{n} < \varepsilon$ as desired.

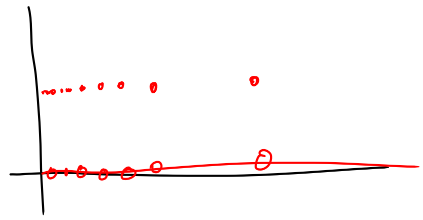
6.2 #2a

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$



It converges to:

$$f(x) = \begin{cases} 1 & \text{if } x = 1/k \text{ for } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$



This is **not** uniform on $\mathbb{R}$, since

$|f_n(x) - f(x)|$ always has some values $= 1$,

no matter how big n is.

So we cannot make $|f_n(x) - f(x)| < \varepsilon$ for any $\varepsilon > 0$, for all $x \in \mathbb{R}$.

6.2 #3b (for g_n)

$$g_n(x) = \frac{x}{1 + x^n} \quad \text{converges to} \quad g(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$g_n(x)$ are all continuous on $[0, \infty)$

but $g(x)$ isn't continuous on $[0, \infty)$,

so the convergence cannot be uniform

by Thm 6.2.6

6.3 #1a $\quad g_n(x) = \dfrac{x^n}{n}$ converges unif to 0 on $[0,1]$.

Proof Let $\varepsilon > 0$ be given, we'll find $N \in \mathbb{N}$ s.t. $\forall x \in [0,1]$

$$n > N \Rightarrow \left| \frac{x^n}{n} - 0 \right| < \varepsilon.$$

$$\left| \frac{x^n}{n} - 0 \right| = \frac{x^n}{n} \leq \frac{1}{n}$$

Let $N > \frac{1}{\varepsilon}$, then if $n > N$ we have

$$\left| \frac{x^n}{n} - 0 \right| \leq \frac{1}{n} < \varepsilon \quad \text{as desired.}$$

g is differentiable since $g(x) = 0$ is constant,

and $g'(x) = 0 \quad \forall x$.