"for all"

$|x|$ is smaller than anything

<u>Thm</u> $x=0$ iff $\forall \varepsilon>0, \quad |x|<\varepsilon$.

$\Longleftrightarrow$

<u>Pf</u> <u>$\Rightarrow$</u> Assume $x=0$, $\boxed{\text{let } \varepsilon>0 \text{ be given.}}$

We'll show $|x|<\varepsilon$.

It's enough to show $0<\varepsilon$,

and this is true since we assumed $\varepsilon>0$.

<u>$\Leftarrow$</u> Assume $\forall \varepsilon>0, \quad |x|<\varepsilon$,

WTS $x=0$.

want to show

<u>For the sake of a contradiction</u>

FSOC, assume $x\neq 0$.

We assumed $\forall \varepsilon, \cdots$

let's try to choose some particular value for ε.

Let $\varepsilon=|x|$.

then $|x|<\varepsilon$ means $|x|<|x|$,

which is a contradiction!

Use the above to prove:

<u>Thm</u> $a = b$ iff $\forall \varepsilon > 0 \quad |a-b| < \varepsilon$.

<u>Pf</u> <u>$\Rightarrow$</u> Assume $a=b$, WTS $\forall \varepsilon > 0$, $|a-b| < \varepsilon$.

Since $a=b$, we have $a-b = 0$,

using $x = a-b$ in the other theorem,

$a-b=0$ means $\forall \varepsilon > 0$, $|a-b| < \varepsilon$.

<u>$\Leftarrow$</u> Assume $\forall \varepsilon > 0$, $|a-b| < \varepsilon$. WTS $a=b$.

Since $|a-b| < \varepsilon$ for all ε, the other theorem

means $a-b=0$. So $a=b$.

Shown.

<u>Completeness</u>

The fundamental property of the set of real #s.

Intuitively: [$\mathbb{R}$ has no "gaps" or missing #s in between others.

[if some real #s approach something, then that thing they approach is also a real #

3, 3.1, 3.14, 3.141, 3.1415, ... ← all rational.

these approach π.

Rational #s can approach an irrational.

So $\mathbb{Q}$ is <u>not</u> complete.

Completeness is defined in terms of <u>bounded</u> sets.

<u>Def</u> a set $A \subseteq \mathbb{R}$ is <u>bounded above</u> if:

$\exists b \in \mathbb{R}$ such that $a \leq b$ $\forall a \in A$.

there exists

such a b is called an upper bound for A.

(there are many upper bounds possible)

a set $A \subseteq \mathbb{R}$ is <u>bounded below</u> if:

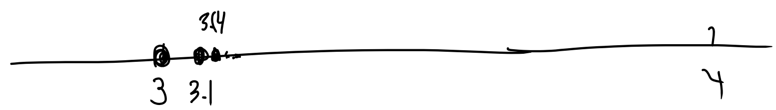
$\exists b \in \mathbb{R}$ such that $a \geq b$ $\forall a \in A$.

This b is called a <u>lower bound</u>.

Def If $A \subseteq \mathbb{R}$ is bounded above, then the supremum of A is the least upper bound of A.

Written $\sup A$

for $A = \{3, 3.1, 3.14, 3.141, \ldots\}$



4 is an upper bound, so A is bounded above.

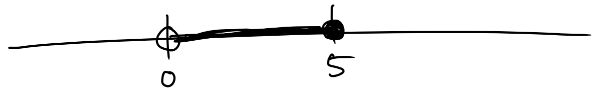
here, $\sup A = \pi$.

Axiom of Completeness If $A \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup A$ exists and $\sup A \in \mathbb{R}$.

The greatest lower bound is the infimum

$\inf A$.

$A = (0, 5]$



$\sup A = 5$

$\inf A = 0$

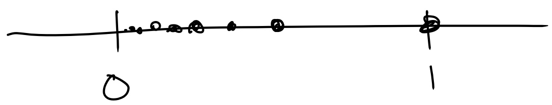
$\mathbb{N}$ natural #s

$\mathbb{N} = \{1, 2, 3, \dots\}$

$B = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$

$\sup B = 1$

$\inf B = 0$



$\{ 1/1, 1/2, 1/3, 1/4, \dots \}$

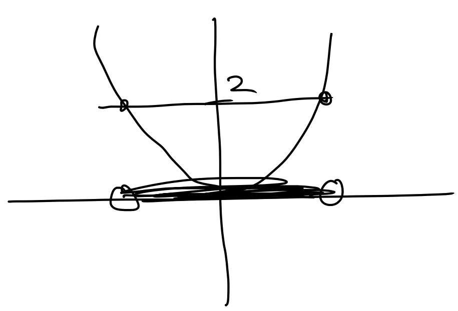
$C = (-\infty, 5)$

$\sup C = 5$

$\inf C$ does not exist.

(C is not bounded below)

$D = \{ x \mid x^2 < 2 \}$



$\sup D = \sqrt{2}$

$\inf D = -\sqrt{2}$