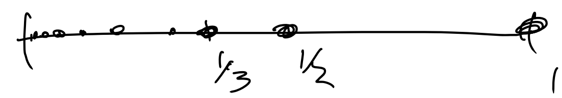
Axiom of Completeness (AoC)

Any bounded-above set has a sup.

least upper bound.

$$\sup \left\{ \tfrac{1}{n} \mid n \in \mathbb{N} \right\} = 1$$
$$\inf \{ \text{-- -- -- . .} \} = 0$$



"sup is the nearest point to the set on the right side"

"inf is ..... left side"

Formalized

if we move the sup even a bit ($\forall \varepsilon > 0$) to the left, then it goes to the left of some point.

Thm let s be an upper bound of A.

then:

$$S = \sup A \iff \forall \varepsilon > 0, \ \exists a \in A \text{ with } s - \varepsilon < a.$$

(iff)

Pf $\Rightarrow$ Assume $s = \sup A$, let $\varepsilon > 0$ be given, we'll find $a \in A$ such that $s - \varepsilon < a$.

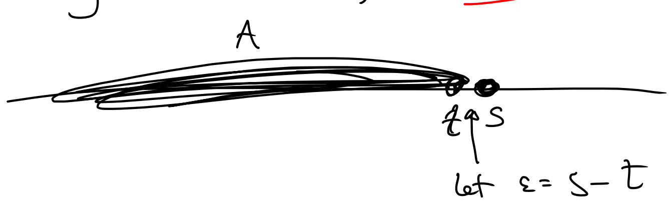
s was the least upper bd, so $s-\varepsilon$ is not an upper bd, so $\exists a \in A$ with $s - \varepsilon < a$. Shown!

$\Leftarrow$ Assume: $\forall \varepsilon > 0$ $\exists a \in A$ with $s - \varepsilon < a$.

We must prove $s = \sup A$.

We assume it's an upper bd, so we need only show it's the least upper bd.

FSOC assume there is a lesser upper bd, say $t < s$, t is an upper bound.



Let $\varepsilon = s - t$, then $\exists a \in A$ s.t.

$$s - (s - t) < a$$

so

$$s - s + t < a$$

so

$$t < a.$$

So t is not an upper bd of A. $\Rightarrow\Leftarrow$

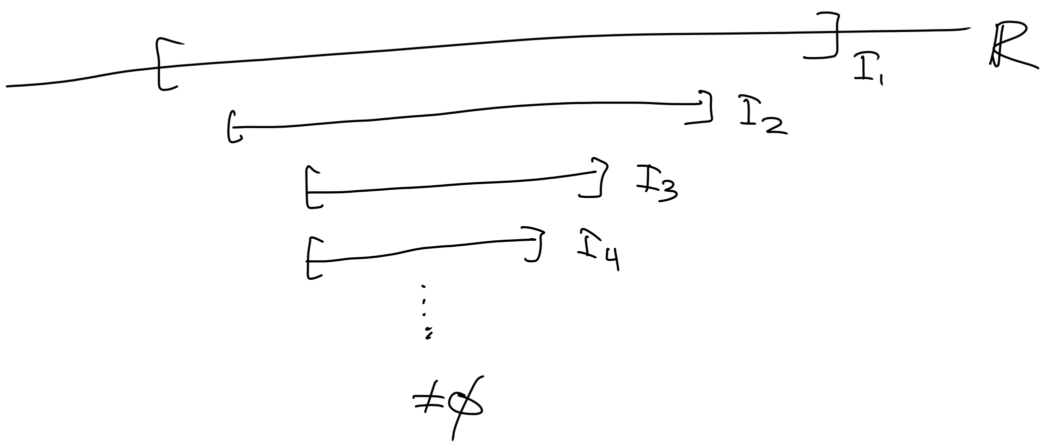
Consequences of Completeness

Thm The Nested Intervals Property (NIP)

Let $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$,

assume $I_{n+1} \subseteq I_n$ for all n.

Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

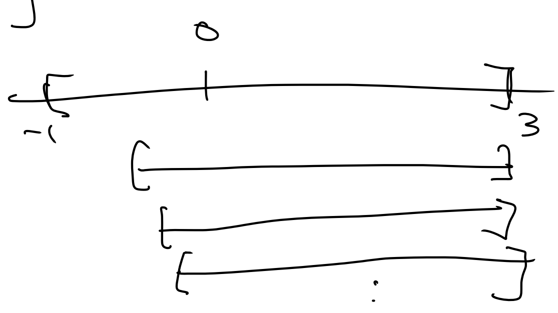


Ex $I_n = [-1/n, 3]$

$I_1 = [-1, 3]$

$I_2 = [-1/2, 3]$

$\vdots$



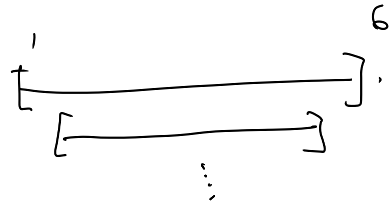
Then in this case

$$\bigcap_{n=1}^{\infty} I_n = [0,3] \neq \emptyset.$$

$$I_n = [2 - 1/n,\ 5 + 1/n]$$

$$I_1 = [2-1,\ 5+1]$$

$$I_2 = [2-1/2,\ 5+1/2]$$



$$\bigcap_{n=1}^{\infty} I_n = [2,5]$$

Nested closed intervals have nonempty intersection

Can you make examples where:

- Nested closed intervals have intersection of 1 point
- .. - - - - - - - - - - .. an open interval
- Nested but not closed, with empty intersection
- Closed but not nested, - - - - - -

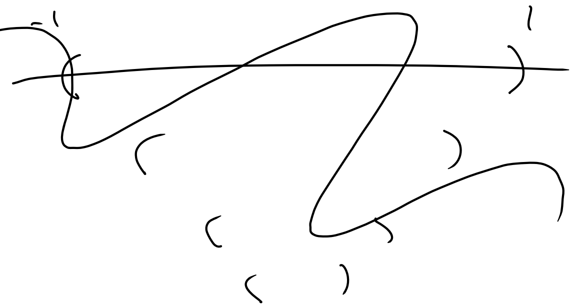
- Nested closed intervals have intersection of 1 point

$$\bigcap_{n=1}^{\infty} [-1/n, 1/n] = \{0\}$$

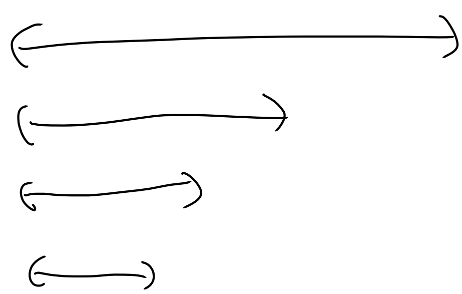
- ~~an open interval~~

- Nested but not closed, with empty intersection

$\cancel{(-1/n, 1/n)}$



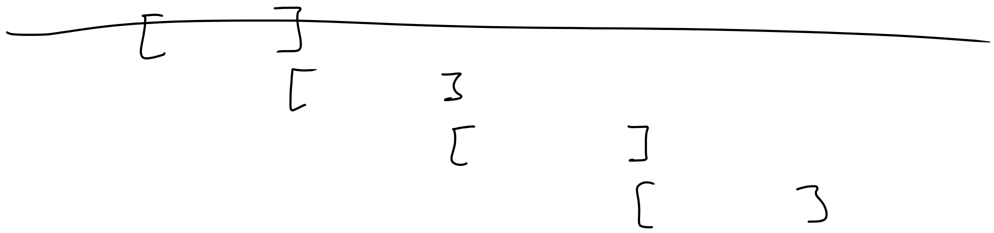
$I_n = (0, 1/n)$



Then $\bigcap_{n=1}^{\infty} I_n = \emptyset$

- Closed but not nested,

$[n, n+1]$



$$\bigcap [n, n+1] = \emptyset.$$