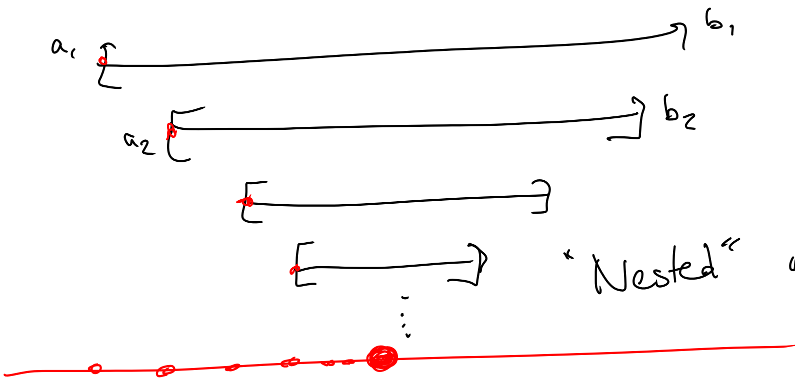
NIP

Let $I_n = [a_n, b_n]$ be nested closed intervals. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.



"Nested" means:

$$a_1 \le a_2 \le a_3 \le \dots \le b_3 \le b_2 \le b_1$$

Proof of NIP

We need to find a point inside all I_n's.

[We'll use AoC: any bounded set has a sup]

Let $a = \sup\{a_n\}$.

(This exists by AoC, $\{a_n\}$ is bounded above by b_1)

We'll show that $a \in \bigcap_{n=1}^{\infty} I_n$.

enough to show $a_n \le a \le b_n$ for every n.

Let $n \in \mathbb{N}$.

$a_1 \le a_2 \le a_3 \le \dots \le \quad \le \dots \le b_3 \le b_2 \le b_1$

a is the least upper bd of $\{a_n\}$

Show $a \geq a_n$

use "upper bound"

Since a is an upper bound for $\{a_n\}$, we know

$$a \geq a_n$$

Show $a \leq b_n$

use "least"

b_n is an upper bound for the a's, and a is the **least** upper bound, so

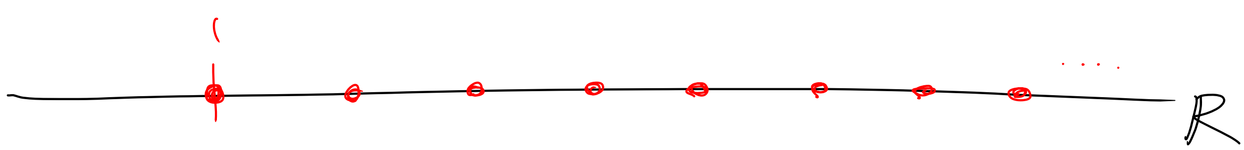
$$a \leq b_n.$$

Shown

Another consequence of AoC:

Archimedean Property

relationship between $\mathbb{R}$ & $\mathbb{N}$



Arch. Prop Big Version For any $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ with $n > x$.

numbers

Of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude.

Archimedes, "On the Sphere and Cylinder", 225BC

No real # is bigger than every $\mathbb{N}$-number.

Thm $\mathbb{N}$ is not bounded above. (Arch Prop, big version)

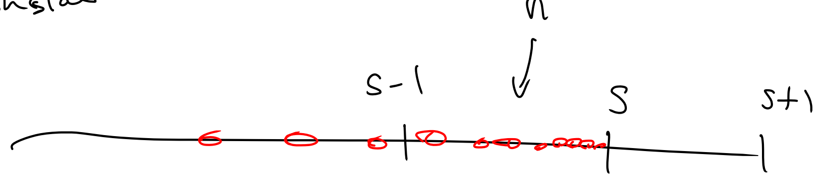
Pf (we'll use AoC)

FSOC, assume $\mathbb{N}$ is bounded above,

then by A.C, $\mathbb{N}$ has a sup.

let $s = \sup \mathbb{N}$, so $s \in \mathbb{R}$

Consider $s-1 \in \mathbb{R}$.



Since $s = \sup \mathbb{N}$, and $s-1 < s$,

$s-1$ is not an upper bound for $\mathbb{N}$,
so $\exists n \in \mathbb{N}$ with

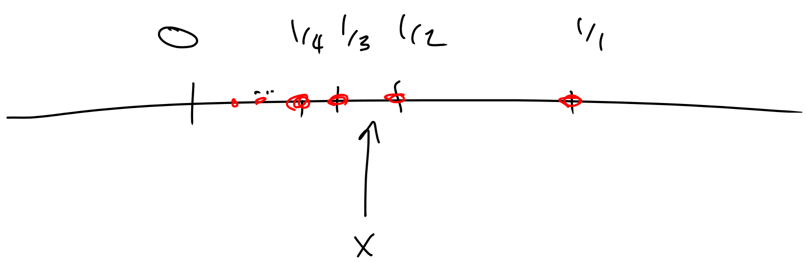
$$s-1 < n \le s$$

add 1 $\longrightarrow$ $$s < n+1 \le s+1$$

s is an upper bound for all $\mathbb{N}$,
but $n+1$ is a natural #, and $s < n+1$.

$\rightarrow\leftarrow$.

Small version is about fractions like $\frac{1}{n}$.



__Arch. Prop. (small)__ For any $x > 0$,
$\exists n \in \mathbb{N}$ s.t. $1/n < x$.

__Pf__ (Take reciprocals, use Big one)
Let $x > 0$. Then $\frac{1}{x} > 0$, then
by the big version, $\exists n \in \mathbb{N}$ s.t.

$n > \frac{1}{x}$. Then $xn > 1$

Then $x > 1/n$ Shown.