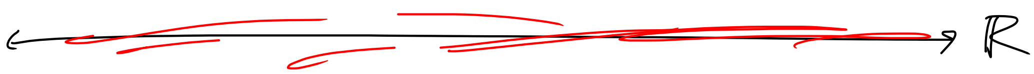
How does $\mathbb{Q}$ sit inside $\mathbb{R}$?



$\mathbb{Q}$ sits evenly dispersed throughout $\mathbb{R}$.

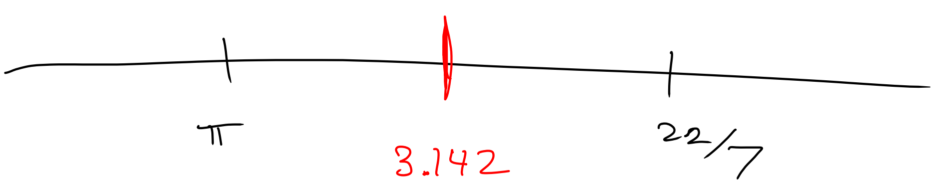
$\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$

<u>Thm</u> For any $a, b \in \mathbb{R}$ with $a < b$,
$\exists\, q \in \mathbb{Q}$ with $a < q < b$.

<u>Example</u> π and $\frac{22}{7}$

$$\pi = 3.14159\ldots$$

$$\frac{22}{7} = 3.14285\ldots$$



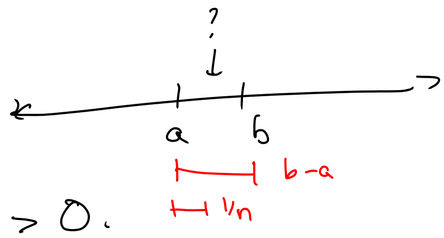
$$3.142 = \frac{3142}{1000}$$

Can we construct this ↑ # without looking at the digits?

We'll use arch. prop.

<u>Thm</u> If $a<b$, then $\exists q \in \mathbb{Q}$ with

$$a < q < b.$$

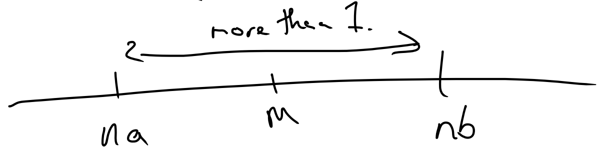


<u>Pf</u> $b-a \in \mathbb{R}$ and $b-a>0$.

Then by Arch. prop (small), $\exists n \in \mathbb{N}$

where $\frac{1}{n} < b-a$

So $1 < nb - na$



So there is an integer in between;

say $na < m < nb$

divide by n:

$$a < \frac{m}{n} < b$$

Shown!

$4 = 4.0000$

$4.00111\ldots$

$4.000112123123412345\ldots$

Cardinality of $\mathbb{R}$

This is about the size of $\mathbb{R}$.

Cardinality means the # of elements.

The cardinality of $\{5,6,7,8\}$ is 4.

card. of $\mathbb{N}$ is infinite.

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$

Cantor in 1870s investigated the cardinalities of various infinite sets.

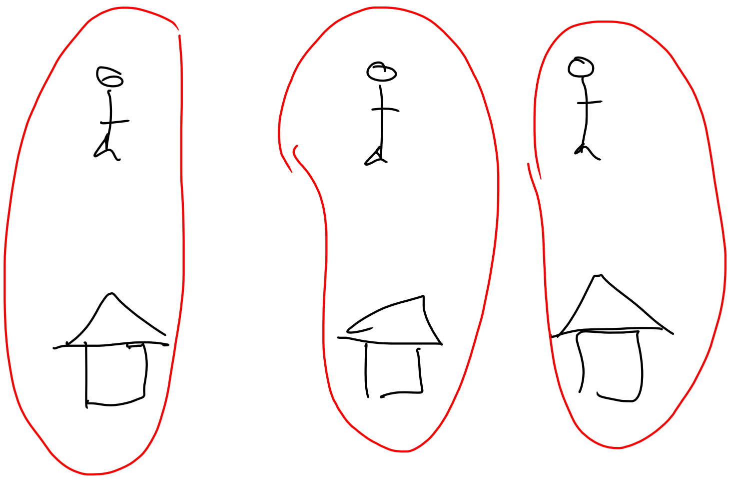
Cantor discovered: There are different sizes of ∞.

~1870s, mathematicians didn't like ∞.

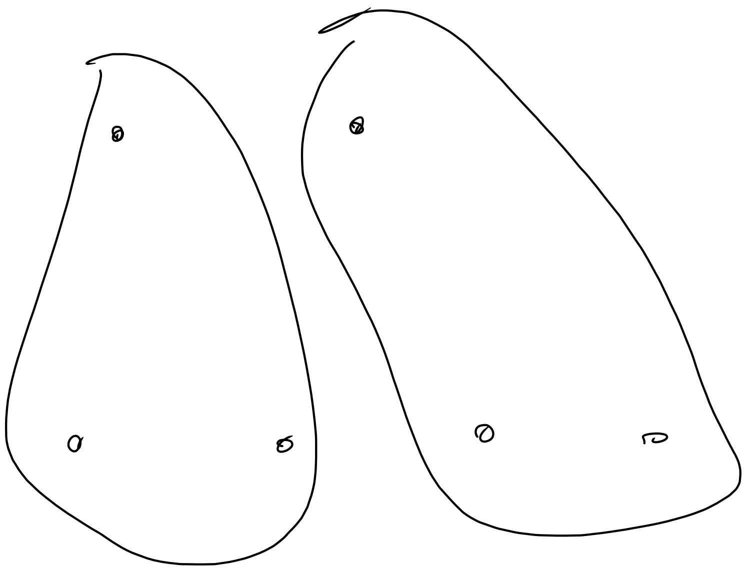
$\lim_{x \to \infty} \ldots$

This is philosophy or theology

"transfinite set theory"



We'll measure sizes by matching up elements.



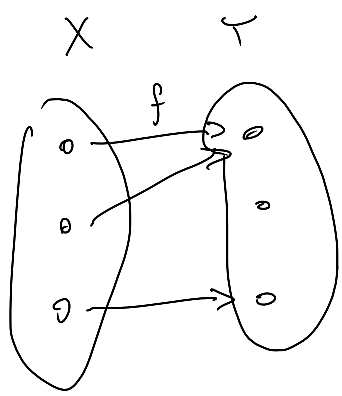
Given sets X & Y,

we build a function

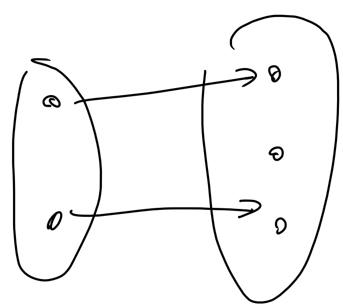
$$\underset{\text{domain}}{f: X} \to \underset{\text{codomain}}{Y}$$

X & Y have the same size if
f is onto and 1-to 1.
↑ surjective ↑ injective.

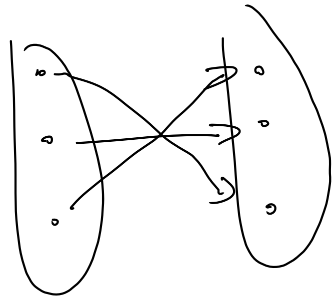
onto & 1 to 1 is <u>bijective</u>



this is not 1-1,
not onto.
∴ f is not a bijection.



it is 1-1
not onto



it is 1-1 and onto.
it is a bijection

Def Two sets A & B have the same cardinality when $\exists$ a bijection $f: A \to B$.

Ex

$A = \{1, 3, 5, 7, \ldots\}$

$B = \{2, 4, 6, 8, \ldots\}$

We want a bijection $f: A \to B$

$f(x) = x + 1$

Ex

$A = \{1, 2, 3, 4, \ldots\}$

$B = \{10001, 10002, 10003, \ldots\}$

$f(x) = x + 10{,}000$