
Michael Kinyon @ProfKinyon · 22m

I remember when the Queen knighted my friend Jective and now he is onto everything.

2 5 32

Cardinality

Write $A \sim B$ when A & B have same cardinality.

i.e. $\exists\, f : A \to B$ which is a bijection.

$$\{1, 3, 5, 7, \dots\} \sim \{2, 4, 6, 8, \dots\}$$

using $f(x) = x + 1$.

$$\underbrace{\{1, 2, 3, \dots\} \sim \{101, 102, 103, \dots\}}_{f(x) = x + 100}$$

means that

$\infty + 100 = \infty$

Weird: we can have $A \subset B$ with $A \sim B$.

$\#\mathbb{N} + 100 = \#\mathbb{N}$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

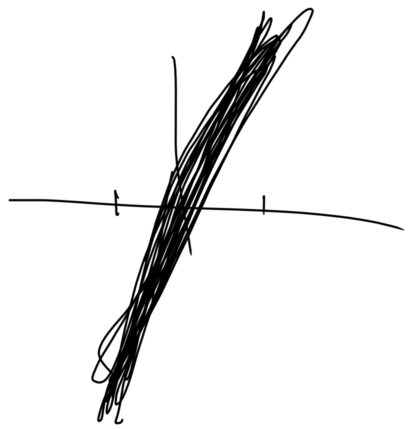
$$2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

$f(x) = 2x$ is a bijection

so $\mathbb{Z} \sim 2\mathbb{Z}$.

$[-1, 1] \sim [-100, 100]$

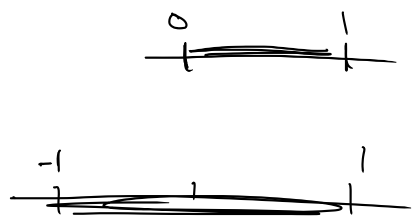
using the function $f(x) = 100x$



$[99, 101]$

$[0,1]$ vs $[-1,1]$

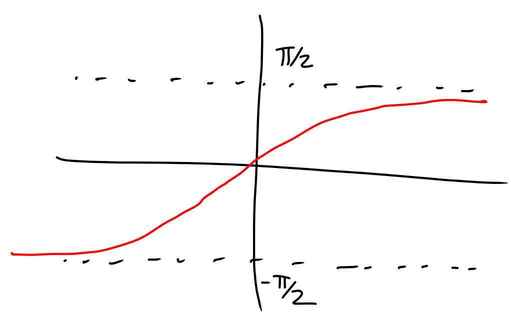
$f(x) = 2x - 1$

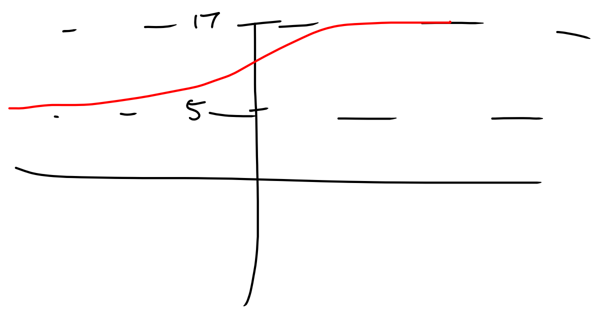


so $[0,1] \sim [-1,1]$

$\mathbb{R} \sim (-\pi/2, \pi/2)$

using: $f(x) = \arctan x$





We say any set A is <u>countably infinite</u> ("countable") when $A \sim \mathbb{N}$.

(This is the smallest size of infinite set)

If A is infinite but $A \nsim \mathbb{N}$, then it is <u>uncountable</u>.

Countable means you can make a bijection

$$\begin{array}{l} \mathbb{N} = \{1, 2, 3, \dots\} \\ \qquad\ \ \updownarrow\ \ \updownarrow\ \ \updownarrow \\ A = \{a_1, a_2, a_3, \dots\} \end{array}$$

<u>Countable</u> means infinite, but you can write them in a list. "enumerable"

$\mathbb{Z}$ is countable:

we can list them like:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

we listed them, so $\mathbb{Z}$ is countable.

How bout $\mathbb{Q}$?

$$\mathbb{Q} = \{0, \underbrace{\tfrac{1}{1}, \tfrac{-1}{1}}_{\text{num \& denom sum to 2}}, \underbrace{\tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{2}{1}, \tfrac{-2}{1}}_{\text{sum to 3}}, \underbrace{\tfrac{1}{3}, \tfrac{-1}{3}, \cancel{\tfrac{2}{2}}, \cancel{\tfrac{-2}{2}}, \tfrac{3}{1}}_{\text{sum to 4.}}, \tfrac{-3}{1}, \ldots\}$$

So $\mathbb{Q}$ is countable.

(Cantor)

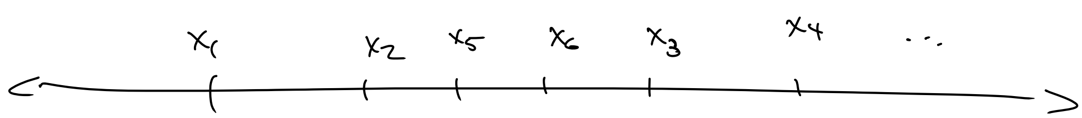
Thm $\mathbb{R}$ is uncountable

PF FSOC assume $\mathbb{R}$ is countable,

i.e.

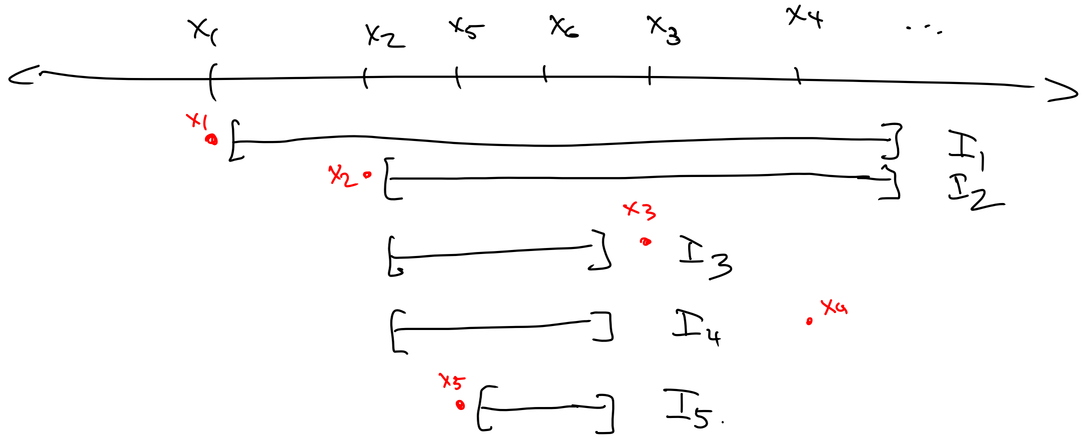
$$\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\}$$

We'll end up showing this list is not exhaustive. i.e. $\exists$ some $\mathbb{R}$ # not in the list.



Make nested closed intervals so that

I_n excludes x_n.



So $x_n \notin I_n$ for every n.

But NIP says: $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$

So $\exists x \in \mathbb{R}$ s.t. $x \in I_n$ for every n.

but $x_n \notin I_n$, so this x has

$x \neq x_n$ for any n.

So x is different from all the x_n's

$\rightarrow\leftarrow$ $\mathbb{R} = \{x_1, x_2, \ldots\}$

In 1890s Cantor's "diagonalization proof"

To get a contradiction, assume

$$\mathbb{R} = \{x_1, x_2, x_3, \dots\}$$

we'll find some real # not in the list.

Write them as decimals:

$$x_1 = 1.0000000\dots$$
$$x_2 = 3.14159\dots$$
$$x_3 = 8.675309\dots$$
$$x_4 = 7.333333\dots$$
$$x_5 = 0.000000$$
$$\vdots$$

A number not in the list

$$x = 2.2841\dots$$

this is different from x_n in digit n.

x_{9981}