$x_1 = 1.00000\ldots$

$x_2 = 0.50000\ldots$

$x_3 = 3.14152\ldots$

$x_4 = 0.63499\ldots$

$x_5 = 4.75319\ldots$

$x_6 = 8.67530\ldots$

No listing can include all real #s.

$\mathbb{R}$ is <u>uncountable</u>

$x = 2.65521\ldots$

So $\#\mathbb{N} < \#\mathbb{R}$

$\aleph_0 \qquad \aleph_1$

if $\#A = n$, then $\#\mathcal{P}(A) = 2^n$

Cantor's diagonal argument can show

$\#S < \#\mathcal{P}(S)$.

So we can make many large cardinalities

$\#\mathbb{R} < \#\mathcal{P}(\mathbb{R}) < \#\mathcal{P}(\mathcal{P}(\mathbb{R})) < \ldots$

$\aleph_2 \qquad \aleph_3$

there's even a $\aleph_{\aleph_0}$

$\#\mathbb{N} < \#\mathbb{R}$

Cantor asked is there a set in between?

The Continuum Hypothesis (Cantor, 1878)

There is no set $S$ with

CH

$$\#\mathbb{N} < \#S < \#\mathbb{R}$$

By 1900 this was a big deal.

In 1940, Gödel couldn't prove it, but showed CH will never lead to a contradiction in set theory.

In 1963, Cohen showed CH will not lead to a contradiction if it is false.

In set theory, CH can't be proven true or false

it is <u>undecideable</u>

# Sequences & Series

A **sequence** is an infinite list

$(a_1, a_2, a_3, \dots) = (a_n)$ (may or may not have a limit)

A **series** is an infinite sum

$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$ (may or may not converge)

---

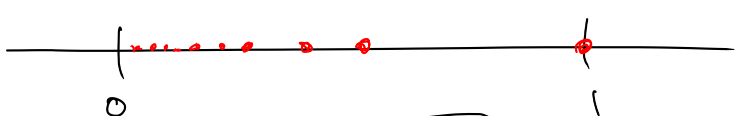
$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) = (\frac{1}{n})$

$(1, 2, 3, 4, \dots) = (n)$

or $(a_n) = \left(\frac{3+\sqrt{n}}{4n^2}\right)$

---

$(1, \frac{1}{2}, \frac{1}{3}, \dots)$ converges to $0$




we write $\boxed{\lim \frac{1}{n} = 0}$.

$\lim\limits_{n \to \infty}$

What does it really mean?

" $\frac{1}{n}$ gets really really close to 0 when n is really big"

$\frac{1}{n} < \varepsilon \quad \forall \varepsilon > 0$ when n is big enough.

$\forall \varepsilon > 0$, when n is big enough, $\frac{1}{n} < \varepsilon$.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

if $n > N$, then $\frac{1}{n} < \varepsilon$.

$$\left[ \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \text{ s.t. } \frac{1}{N} < \varepsilon \right]$$

---

**Def** Let $(a_n)$ be a sequence. We say $(a_n)$ converges to $a \in \mathbb{R}$ when:

memorize this

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

dist from $a_n$ to $a$.

$$n > N \implies |a_n - a| < \varepsilon$$

depends on $\varepsilon$ (usually)

In this case we write $\lim a_n = a$.

**Ex** Show $\lim \frac{1}{n} = 0$.

[WTS $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.
$n > N \Rightarrow |a_n - a| < \varepsilon$]

**Pf**

① [Let $\varepsilon > 0$ be given, we'll find $N \in \mathbb{N}$
s.t. $n > N \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$.

② Choose $N > \frac{1}{\varepsilon}$ ← hard part

③ Now let $n > N$, and we have:

$$\left|\frac{1}{n} - 0\right| = \left|1/n\right| = 1/n < 1/N < \frac{1}{1/\varepsilon} = \varepsilon$$

as desired.

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**The Prestige**

① The Pledge: shows the object.

② The Turn: makes the ordinary object do something extraordinary.

③ The Prestige: shows the object again as it was.