$\lim a_n = a$ or $a_n \to a$

means: $\forall \varepsilon > 0 \ \exists n \in \mathbb{N}$ s.t.

$n > N \implies |a_n - a| < \varepsilon.$

when n is really big

a_n & a are really close together

Ex Show $\lim \frac{n+5}{n} = 1$

Pf ① Let $\varepsilon > 0$ be given. We'll find N s.t.

$n > N \implies \left| \frac{n+5}{n} - 1 \right| < \varepsilon.$

② Let $N > 5/\varepsilon$.

simplify

$\left| \frac{n+5}{n} - 1 \right| = \left| \frac{n+5}{n} - \frac{n}{n} \right|$

$= \left| \frac{n+5-n}{n} \right|$

$= \left| \frac{5}{n} \right| = \frac{5}{n}$

we need $\frac{5}{n} < \varepsilon$

solve for n:

$5 < n \cdot \varepsilon$

$n > 5/\varepsilon$

③ Then if $n > N$, we have:

$\left| \frac{n+5}{n} - 1 \right| = \frac{5}{n} < \frac{5}{N} < \frac{5}{5/\varepsilon} = \varepsilon$

so $\left| \frac{n+5}{n} - 1 \right| < \varepsilon$ as desired.

Show $\quad 2+\frac{1}{\sqrt{n}} \to 2$

$\frac{n}{5+n} \to 1$

$2+\frac{1}{\sqrt{n}} \to 2$

PF $\quad$ let $\varepsilon>0$ be given, we'll find $N \in \mathbb{N}$ s.t. $\quad n>N \Rightarrow \left|2+\frac{1}{\sqrt{n}}-2\right| < \varepsilon$

$\left|2+\frac{1}{\sqrt{n}}-2\right| = \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}$

we need $\frac{1}{\sqrt{n}} < \varepsilon$

$\frac{1}{n} < \varepsilon^2 \qquad n > \frac{1}{\varepsilon^2}$

Let $N > \frac{1}{\varepsilon^2}$. Then if $n > N$, we have

$\left|2+\frac{1}{\sqrt{n}}-2\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon$ as desired.

$\frac{n}{n+5} \to 1$

PF $\quad$ let $\varepsilon>0$ be given, we'll find $N \in \mathbb{N}$ s.t. $\quad n>N \Rightarrow \left|\frac{n}{n+5}-1\right| < \varepsilon$.

$\left|\frac{n}{n+5}-1\right| = \left|\frac{n}{n+5}-\frac{n+5}{n+5}\right| = \left|\frac{n-(n+5)}{n+5}\right| = \left|\frac{-5}{n+5}\right|$

$= \frac{5}{n+5} \qquad$ want $\frac{5}{n+5} < \varepsilon \qquad 5 < \varepsilon(n+5)$

$\frac{5}{\varepsilon} - 5 < n$

Let $N > \frac{5}{\varepsilon} - 5$. Then let $n > N$, we have:

$$\left|\frac{n}{n+5} - 1\right| = \frac{5}{n+5} < \frac{5}{N+5} < \frac{5}{5/\varepsilon - 5 + 5} = \frac{5}{5/\varepsilon}$$
$$= \varepsilon$$

Shown.

Show $\frac{n^2}{n^3+2} \to 0$

<u>Pf</u> Let $\varepsilon > 0$, we'll find $N \in \mathbb{N}$ s.t.

$$n > N \Rightarrow \left|\frac{n^2}{n^3+2} - 0\right| < \varepsilon$$

simplify

$$\left|\frac{n^2}{n^3+2} - 0\right| = \left|\frac{n^2}{n^3+2}\right| = \frac{n^2}{n^3+2}$$

we need $\frac{n^2}{n^3+2} < \varepsilon$

~~It's enough~~ if $\frac{n^2}{n^3} < \varepsilon$

$$\Downarrow$$

$\frac{1}{n} < \varepsilon$, so $n > \frac{1}{\varepsilon}$,

$$\left|\frac{n^2}{n^3+2}\right| = \frac{n^2}{n^3+2} < \frac{n^2}{n^3} = \frac{1}{n}$$

so $n > 1/\varepsilon$

In the simplifying step, we can use $=$'s or $<$ (not $>$)

Let $N > 1/\varepsilon$. Then if $n > N$, we have:

$$\left|\frac{n^2}{n^3+2} - 0\right| = \frac{n^2}{n^3+2} < \frac{n^2}{n^3} = \frac{1}{n} < \frac{1}{N} < \frac{1}{1/\varepsilon} = \varepsilon$$

Shown!