When simplifying we can use $=$ or $<$.

Ex) $\dfrac{2n^2+7n-1}{3n^2-1} \to \dfrac{2}{3}$

Pf Let $\varepsilon>0$ be given, we'll find some $N \in \mathbb{N}$ s.t.

$$n > N \implies \left| \frac{2n^2+7n-1}{3n^2-1} - \frac{2}{3} \right| < \varepsilon.$$

$$\left| \frac{2n^2+7n-1}{3n^2-1} - \frac{2}{3} \right| = \left| \frac{3(2n^2+7n-1)}{3(3n^2-1)} - \frac{2(3n^2-1)}{3(3n^2-1)} \right|$$

$$= \left| \frac{\cancel{6n^2}+21n-3-\cancel{6n^2}+2}{3(3n^2-1)} \right|$$

$$= \left| \frac{21n-1}{9n^2-3} \right| = \frac{21n-1}{9n^2-3} < \frac{21n}{9n^2-3}$$

$$\frac{21n}{9n^2-3} < \frac{21n}{9n^2-3n^2} = \frac{21n}{6n^2} = \frac{21}{6n} = \frac{7}{2n}$$

We need $\dfrac{7}{2n} < \varepsilon$, so $\boxed{n > \dfrac{7}{2\varepsilon}}$

Typical "<" tricks

- Throw away neg. ~~constant~~ term in numerator,
- Throw .. .. pos. .. .. .. denominator
- Change pos. constant in numerator to n's
- Change neg .. .. .. denom to n's.

$3n^2+4 < 3n^2+4n^2$

Let $N > 7/2\varepsilon$. Then if $n > N$, we have:

$$\left|\frac{2n^2+7n-1}{3n^2-1} - \frac{2}{3}\right| < \frac{7}{2n} < \frac{7}{2N} < \frac{7}{2\cdot\frac{7}{2\varepsilon}} = \varepsilon$$

as desired.

Comp: $\frac{n}{n+1} \to 1$

Try $\frac{n+5}{2n^2-n+1} \to 0$

$\frac{n+\sin n}{2n} \to \frac{1}{2}$

$$\left|\frac{n+5}{2n^2-n+1}\right| = \frac{n+5}{2n^2-n+1} < \frac{n+5n}{2n^2-n+1} < \frac{6n}{2n^2-n}$$

$$< \frac{6}{2n-1} < \frac{6}{2n-n} = \frac{6}{n}$$

$\frac{6}{n} < \varepsilon$ $\boxed{n > \frac{6}{\varepsilon}}$

$$\left|\frac{n+\sin n}{2n} - \frac{1}{2}\right| = \left|\frac{n+\sin n}{2n} - \frac{n}{2n}\right|$$

$$= \left|\frac{n+\sin n - n}{2n}\right| = \left|\frac{\sin n}{2n}\right| = \frac{|\sin n|}{2n} \le \frac{1}{2n}$$

$$< \frac{1}{n}$$

$\frac{1}{n} < \varepsilon \qquad n > \frac{1}{\varepsilon}$

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a_n converges to a means:

$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t. $\quad n > N \Rightarrow |a_n - a| < \varepsilon$

$\forall x \in \mathbb{R}, \quad x^2 \geq 0$

$\downarrow \neg$

$\exists x \in \mathbb{R}, \quad x^2 < 0$

a_n doesnot converge to a means:

$\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$,

$\boxed{n > N \Rightarrow |a_n - a| \geq \varepsilon}$

$\neg(p \to q) \equiv p \wedge \neg q$