How to show a sequence does not converge?

$a_n \to a$ $\quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$

$$n > N \Rightarrow |a_n - a| < \varepsilon$$

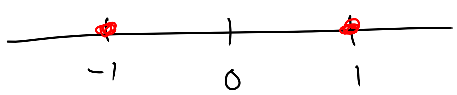
a_n does not converge to a: $\quad \exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \in \mathbb{N}$ s.t.

$$n > N \text{ and } |a_n - a| \geq \varepsilon.$$

" No matter how far down the sequence you go, there's points further where a_n is far from a.

$$(a_n) = (-1)^n = (-1, 1, -1, 1, -1, \dots)$$

Let's show a_n does not converge to 0.



Let $\varepsilon = 1$, let $N \in \mathbb{N}$ be given.

$$|a_n - a| = |(-1)^n - 0| = 1$$

so $|a_n - a| \geq \varepsilon$. for all n.

Thm If (a_n) converges, then the limit is unique.

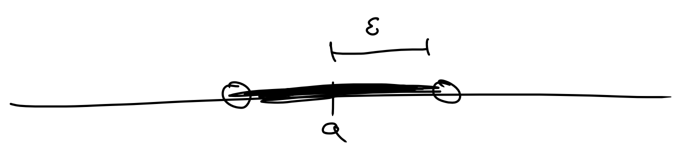
A topological view of convergence.

about shapes & "closeness"

For $a \in \mathbb{R}$, let $V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x-a| < \varepsilon\}$

"the ε-neighborhood of a"

$$V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)$$



Convergence topologically:

$a_n \to a$ means:

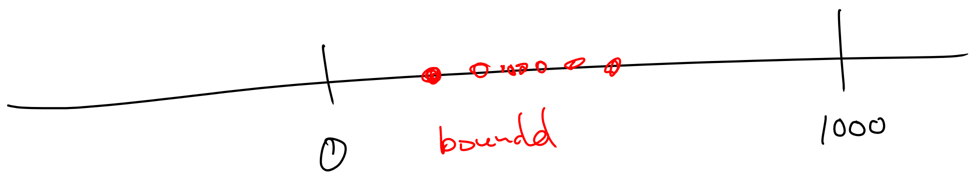
$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ s.t.

$n > N \Rightarrow a_n \in V_\varepsilon(a)$

Properties of limits

We say (a_n) is <u>bounded</u> when $\exists M \in \mathbb{R}$ s.t.

$$|a_n| \leq M \quad \forall n.$$

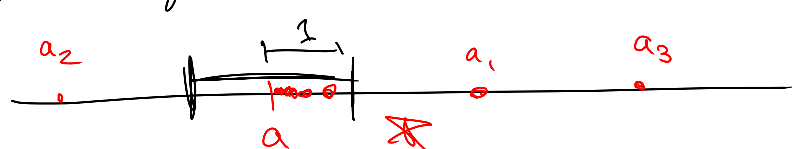


(a_n) is bounded

vs (a_n) converges.

$(a_n) = (-1)^n$ is bounded, but doesn't converge.

Thm If (a_n) converges, then (a_n) is bounded.



all terms in here past from N.

Pf Say $a_n \to a$.

Let $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ s.t.

$$n > N \Rightarrow |a_n - a| < 1.$$

looks like it's bounded by $|a|+1$
but could be bigger for terms before a_N

Let $M = \max(|a|+1, |a_1|, |a_2|, \ldots, |a_N|)$

then $|a_n| \leq M$ for all n. Shown.

convergent $\Rightarrow$ bounded

bounded $\not\Rightarrow$ convergent.

Limit Rules

Thm Let $a_n \to a$ and $b_n \to b$ and $c \in \mathbb{R}$

i) $\lim c a_n = c \lim a_n = ca.$

ii) $\lim (a_n + b_n) = a + b$

iii) $\lim (a_n b_n) = (\lim a_n)(\lim b_n) = ab$

iv) $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ whenever $b \neq 0.$

Pf of #i   WTS $ca_n \to ca$

Let $\varepsilon > 0$ be given, we'll find $N \in \mathbb{N}$ s.t.

$$n > N \Rightarrow |ca_n - ca| < \varepsilon$$

$$|ca_n - ca| = |c(a_n - a)| = |c|\,|a_n - a|$$

want $|c|\underbrace{|a_n - a|} < \varepsilon$

since $a_n \to a$, this part is as small as we want.

I want it to be smaller than $\varepsilon/|c|$

Let N be so big that $n > N \Rightarrow |a_n - a| < \varepsilon/|c|$.

then let $n > N$, we have:

$$|ca_n - ca| = |c|\,|a_n - a| < |c| \cdot \varepsilon/|c| = \varepsilon$$ as desired!