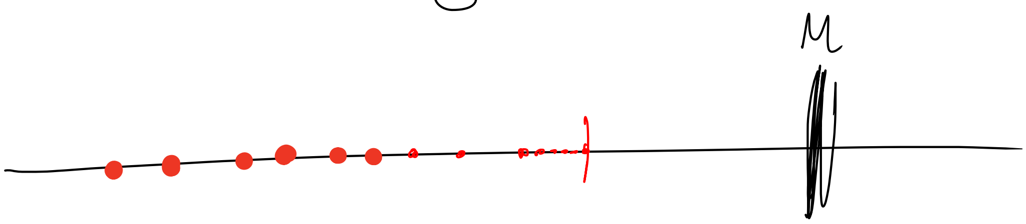
The Monotone Convergence Theorem

Bounded $\not\Rightarrow$ Convergent

$(-1)^n$ is bounded but doesn't converge
it diverges

Bounded & increasing $\Rightarrow$ convergent



Def (a_n) is increasing when $a_n \leq a_{n+1}$ $\forall n$.

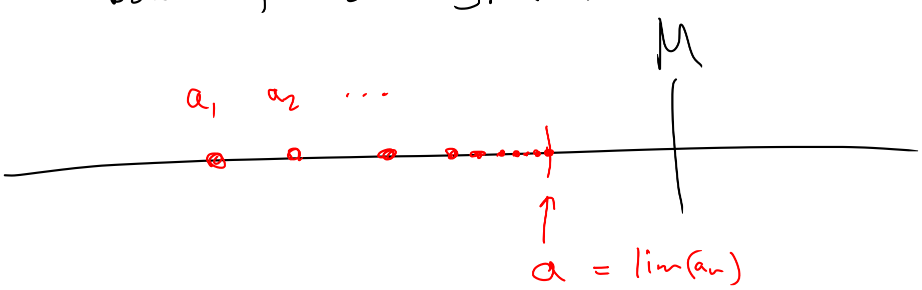
(a_n) is decreasing when $a_n \geq a_{n+1}$ $\forall n$.

(a_n) is monotone when it is increasing
or decreasing.

Monotone Convergence Theorem

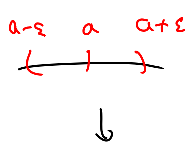
If (a_n) is monotone & bounded then it converges.

Pf Assume without loss of generality that (a_n) is increasing. Assume (a_n) is bounded, so $\exists M \in \mathbb{R}$ with $|a_n| < M$.



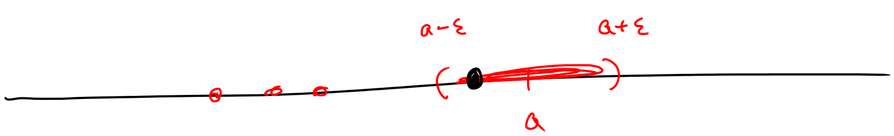
What exactly is a in terms of (a_n)?

Let $a = \sup\{a_n\}$



We'll show $a_n \to a$.

We'll show $\forall \varepsilon > 0$ $\exists N$ s.t. $n > N \Rightarrow a_n \in V_\varepsilon(a)$



Since a is $\sup\{a_n\}$, $\exists$ one point $a_N \in V_\varepsilon(a)$

If $n > N$, a_n is not less than $a-\varepsilon$, since a_n is increasing,

and a_n is not more than $a+\varepsilon$
since a is $\sup\{a_n\}$.

So $\quad a_n \in (a-\varepsilon,\ a+\varepsilon) = V_\varepsilon(a)$.

Shown.

Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{or} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$3 \times (5+2) \div 3 \times 7 + 1$$

add up the real numbers from 0 to 1.

- - - - - - - .5 to 1.

$$\cancel{\int_0^1 x\,dx = 1/2}$$

$\sum_{n=1}^{\infty} a_n$ is defined as the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

$$= \lim \{ a_1, \; a_1+a_2, \; a_1+a_2+a_3, \dots \}$$

the partial sums

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

We'll show it converges using MCT.

Looking at the sequence:

$$S_n = \{ 1, \; 1+1/4, \; 1+1/4+1/9, \dots \}$$

S_n is increasing (so its monotone)

Bounded

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2}$$

$$= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \dots + \frac{1}{n\cdot n}$$

$$< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \dots + \frac{1}{n\cdot(n-1)}$$

$$= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 + 1 - \frac{1}{n}$$

So $S_n < 2 - 1/n < 2$

So S_n is bounded.

So S_n converges, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ exists.}$$

actually, $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

Subsequences

A subsequence of some sequence is a sequence obtained by omitting some terms. (in the same order)

$(a_n) = (1, 2, 3, 4, \ldots)$

Some subsequences: $(2, 4, 6, 8, \ldots)$

$(2, 3, 5, 7, 11, 13, \ldots)$

Not subsequences:

$(2, 1, 4, 3, 5, 6, 7, \ldots)$

$(1, 1, 2, 2, 3, 3, \ldots)$

Def Let (a_n) be a sequence,
and let $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of n's
call it (n_k)

Then a subsequence of (a_n) is a sequence of the form

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots) = (a_{n_k})$$

So if $(a_n) = (1, 2, 3, \ldots)$
and I want terms $5, 10, 15, 20, \ldots$

let $n_k = (5k) = (5, 10, 15, 20, \ldots)$

Then (a_{n_k}) is the terms #5, 10, 15 from a_n.

$$(a_{n_k}) = (5, 10, 15, \ldots)$$

Thm If $(a_n) \to a$ and (a_{n_k}) is a subsequence,

then: $(a_{n_k}) \to a$.