$(a_n) = (2, 4, 6, 8, \ldots)$

$(a_{n_k}) = (4, 8, 12, \ldots)$

this is a subsequence.

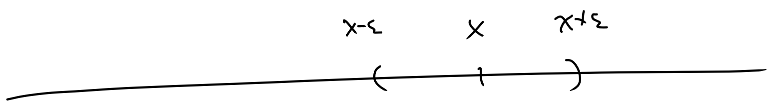
Comp 2022

#22d [2 pts] Prove that if $(x_n) \to x$, then all subsequences of x_n converge to x.

Let (x_{n_k}) be a subsequence. We'll show $(x_{n_k}) \to x$.

Pf Let $\varepsilon > 0$ be given, we'll find N s.t.

$k > N \Rightarrow x_{n_k} \in V_\varepsilon(x)$



We know $\exists N$ s.t. $n > N \Rightarrow x_n \in V_\varepsilon(x)$

so if $k > N$, then x_{n_k} is further along in the sequence than x_N.

So $x_{n_k} \in V_\varepsilon(x)$ as desired.

Thm If $x_n \to x$ and x_{n_k} is a subsequence, then $x_{n_k} \to x$.

Some Contrapositives:

- If any subsequence (x_{n_k}) diverges, then (x_n) diverges.

- If we have 2 subsequences with different limits, then (x_n) diverges.

$(-1)^n$ diverges:

$(-1, 1, -1, 1, -1, 1, \ldots)$

I'll find 2 subseqs with different limits:

$(-1, -1, -1, \ldots) \to -1$

$(1, 1, 1, \ldots) \to 1$

different, so $(-1)^n$ diverges.

Show $n \cdot \underbrace{\sin\left(\frac{\pi n}{2}\right)}$ diverges.

$1, 0, -1, 0, 1, 0, -1, 0, \ldots$

1 2 3 4 5 6 7

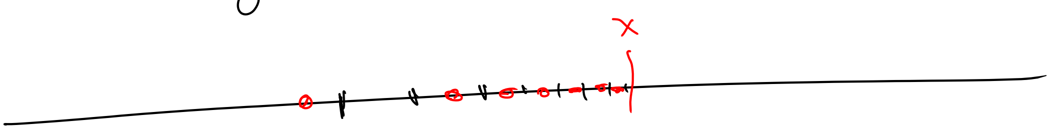
so it's $(1, 0, -3, 0, 5, 0, -7, 0, 9, \ldots)$

$(1, 5, 9, 13, \ldots)$ is a subsequence, which diverges (unbounded),

$(4n-3)$

so the original sequence diverges.

#22e Prove if (x_n) is increasing and has a convergent subsequence, then (x_n) converges.



Pf We'll show (x_n) is bounded.
Since it's increasing, MCT will imply it converges.

Since x_{n_k} converges to x, x_{n_k} is bounded, and since it's increasing,
$x_{n_k} \le x$ for all k.

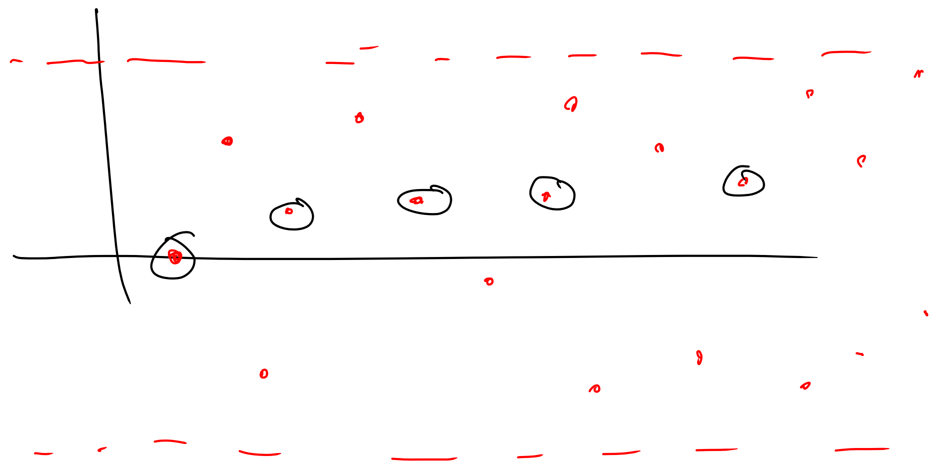
Since x_n is increasing,
each x_n is less or equal to some x_{n_k}.

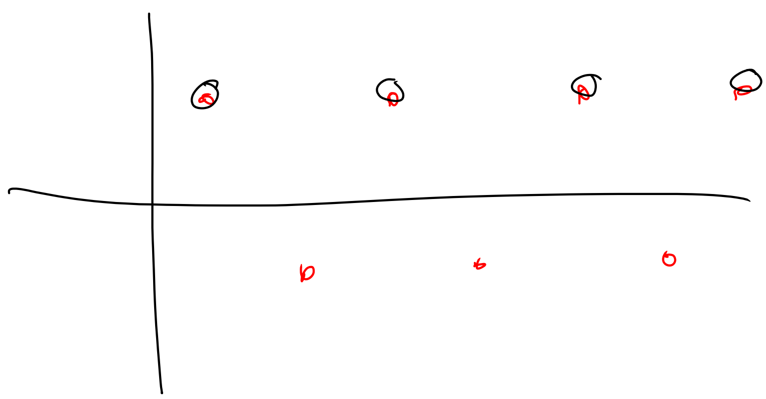
So $x_n \le x_{n_k} \le x$

So x_n is bounded as desired.

Bolzano - Weierstrauss Thm

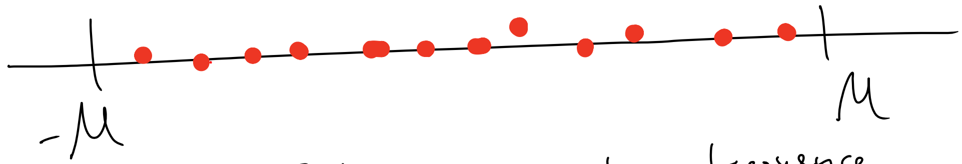
If (a_n) is bounded, then there is a convergent subsequence.



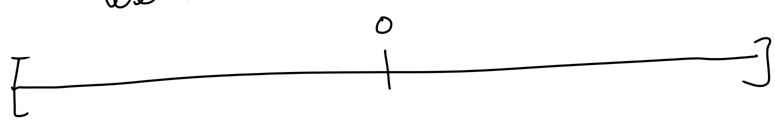


Bolzano - Weierstauss Rap

Proof using NIP

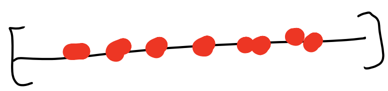


We'll find a convergent subsequence.

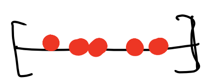


$I_1 = [-M, M]$

Choose intervals so that we always choose a side which contains ∞-ly many terms of the sequence.



I_2

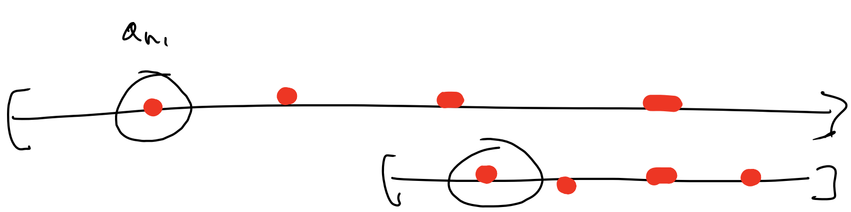


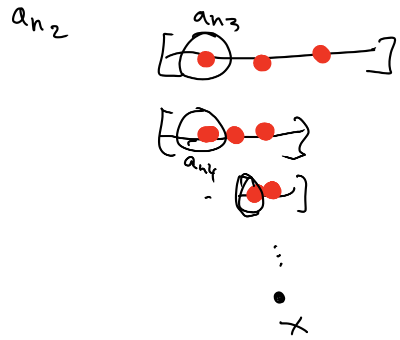
I_3

etc.

We obtain a seq. of nested closed intervals I_n, all contain terms of the sequence,

and $\exists$ a number $x \in I_n$ for all n.





Choose any point

$a_{n_k} \in I_k$.

This is a subsequence which converges to x.