Topology of $\mathbb{R}$

about open & closed sets

(a,b) is an open set - is not closed.

$[a,b]$ is a closed set - is not open

$[a,b)$ is not open or closed

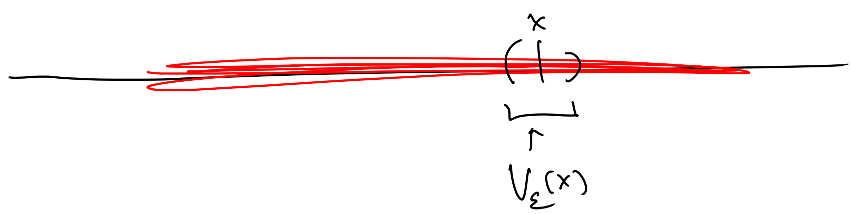
$(-\infty,\infty)=\mathbb{R}$ is open and closed.

"An open set has no boundry points"

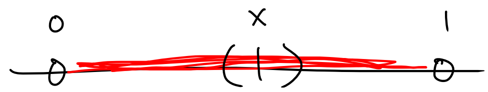
"every point is an interior point"

Def a set $\mathcal{O}\subseteq\mathbb{R}$ is <u>open</u> when:

$$\forall x\in\mathcal{O},\quad \underbrace{\exists\varepsilon>0 \text{ s.t. } V_\varepsilon(x)\subseteq\mathcal{O}.}_{\text{x is surrounded by a nbhd of values in } \mathcal{O}.}$$



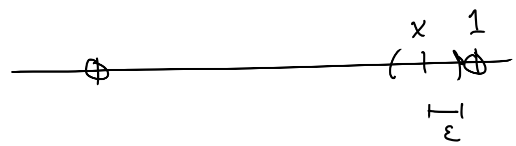
Ex $(0,1)$ is an open set.



Pf Take any $x \in (0,1)$, then

choose

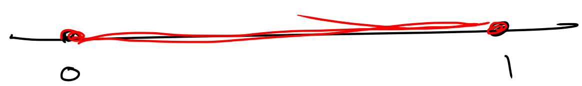
$\varepsilon = \min(x, 1-x)$



Then $V_\varepsilon(x) \subseteq (0,1)$.

Ex $[0,1]$ is not open.

Pf



Take $x = 1$, then every nbhd $V_\varepsilon(x)$ is not a subset of $[0,1]$.

↳ $(1-\varepsilon, 1+\varepsilon)$

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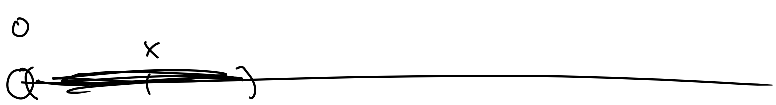
not open → $(0,1]$ if $x=1$, then $V_\varepsilon(x) \not\subseteq (0,1]$.

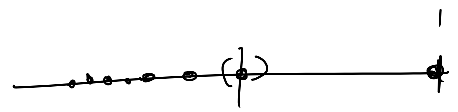
is open → $(0,\infty)$ for $x \in (0,\infty)$ choose $\varepsilon = x$ is open



$(-\infty, \infty)$ is open.

Open or not?

$\{ 1/n \mid n \in \mathbb{N} \}$



not open - any $V_\varepsilon(1)$ goes outside the set.

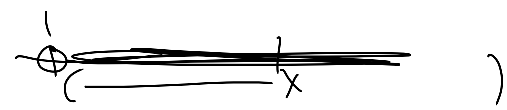
Comp 2022

$\{ x \in \mathbb{R} \mid x > 1 \}$

$\mathbb{Q} \cap [0,1]$

$\mathbb{Q} \cap (0,1)$

$(1, \infty)$ is open



$\varepsilon = x - 1$

$\mathbb{Q} \cap [0,1]$ is not open since

$V_\varepsilon(1)$ includes points more than 1.

So $V_\varepsilon(1) \not\subseteq \mathbb{Q} \cap [0,1]$

$\mathbb{Q} \cap (0,1)$ is not open since

$V_\varepsilon(1/2)$ includes some irrationals,

So $V_\varepsilon(1/2) \not\subseteq \mathbb{Q} \cap (0,1)$.

Same reason: $\mathbb{Q}$ is not open.

$\not{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$ set of irrationals:

is not open.

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not all open sets are just intervals

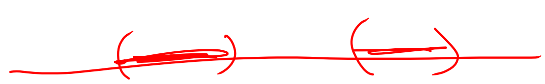
$(0,1) \cup (2,3)$

Generally, an open set of $\mathbb{R}$ looks like some union of open intervals.

Unions and Intersections of open sets.

__Thm__ If A & B are open, then $A \cup B$ is open.

__Pf__



__Pf__ Take $x \in A \cup B$, we'll find some $V_\varepsilon(x) \subseteq A \cup B$.

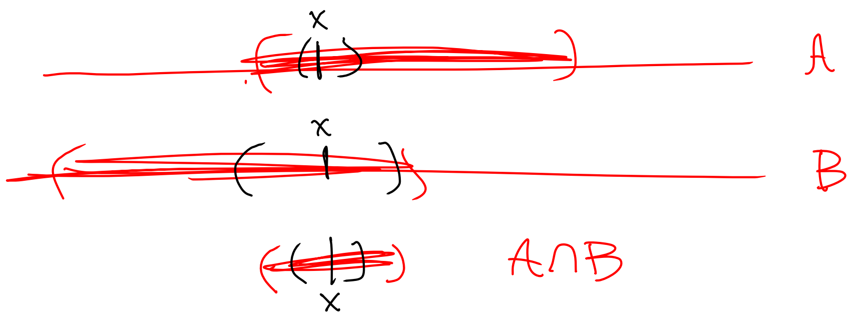
Without loss of generality, assume $x \in A$. (since $x \in A$ or $x \in B$)

Since A is open, there is some $V_\varepsilon(x) \subseteq A$.

since $A \subseteq A \cup B$, this means

$V_\varepsilon(x) \subseteq A \cup B$ Shown.

Thm If A & B are open, then $A \cap B$ is open.



Pf Let $x \in A \cap B$, we'll find a nbhd
$V_\varepsilon(x) \subseteq A \cap B$.

Since $x \in A$, we have a nbhd $V_\alpha(x) \subseteq A$

and since $x \in B$. - - - - - . $V_\beta(x) \subseteq B$.

Let $\varepsilon = \min(\alpha, \beta)$. Then
$V_\varepsilon(x) \subseteq A \cap B$ as desired.

Is $\emptyset$ open?

A is open means: $\underline{\forall x \in A \ \exists \varepsilon \text{ s.t. } V_\varepsilon(x) \subseteq A.}$

if $A = \emptyset$:

$\forall x \in \emptyset, \exists \varepsilon$. - - - $\emptyset$

Vacuously True!

$\forall x \in \breve{\emptyset}$, ... is automatically true

$p \to q$

$\emptyset$ <u>is</u> open