A is Open: $\forall x \in A$, $\exists V_\varepsilon(x) \subseteq A$.

F is closed: F contains all its limit pts.

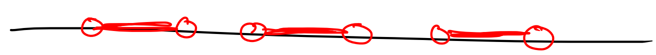
For $A \subseteq \mathbb{R}$, write

A^c is the <u>complement</u>

$$A^c = \mathbb{R} - A$$

<u>Thm</u> The complement of a open is closed.
The complement of a closed is open.

<u>Pf</u> <u>1st</u> an open set looks like:



its complement will look like

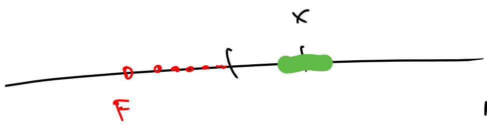
& this is closed.

<u>2nd</u> Let F be closed. WTS F^c is open.

Take $x \in F^c$, WTS $\exists V_\varepsilon(x) \subseteq F^c$.

Since $x \in F^c$, $x \notin F$ since F is closed,

$x \notin F$ means: x is not a limit pt of F,



So $\exists V_\varepsilon(x)$ with

$V_\varepsilon(x)$ contains no point of F.

i.e. every pt of $V_\varepsilon(x)$ is outside F,

so $V_\varepsilon(x) \subseteq F^c$. Shown.

Recall: Any union of open sets is open
Any finite intersection of open sets is open.

Recall: $(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$ De Morgan's Law for sets.

<u>Thm</u> Any intersection of closed sets is closed
Any finite union of closed sets is closed.

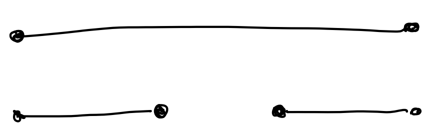
<u>An important wild example closed set.</u>

<u>The Cantor Set</u>

Built in stages:

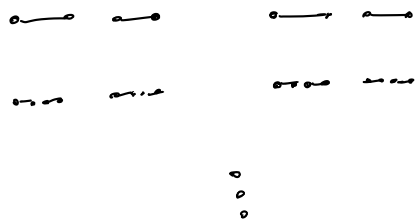
$C_1 = [0,1]$

$C_2 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$



C_3 :

C_4 :



The Cantor set is:

$$C = \bigcap_{i=1}^{\infty} C_i$$

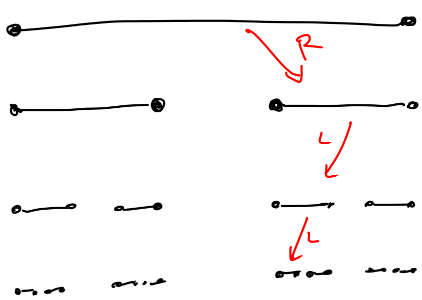
This includes $0, 1, \frac{1}{3}, \frac{2}{3}$, & all other endpoints of intermediate stages.

Properties

C is not empty. C is infinite.

C is closed.

C is uncountable!



Pts in C can be described by lists of L's & R's

Infinite lists.

$x \leftarrow R, L, L, \dots$

If I try to list all elts of C,
it'll look like:

$x_1 = L R L L L R R L \ldots$

$x_2 = L R R L L R L R \ldots$

$x_3 = R R L L R L R R \ldots$

$\vdots$

I can can construct pts missing from the list:

$x = R L R \ldots$

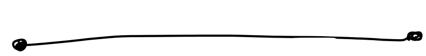
So no list can include all pts of C.

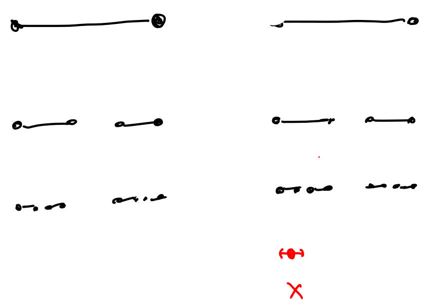
C is <u>uncountable</u>.

$0.02002000200002\ldots$ in base 3 is irrational, in C.

<u>C is not open</u>: Take $x \in C$

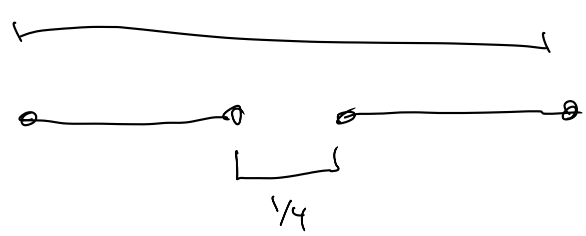
is $V_\varepsilon(x) \subseteq C$? No – C contains no intervals, since intervals always have their middles removed.

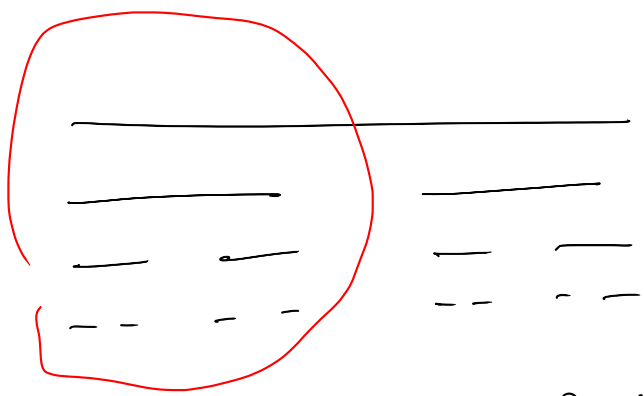




a self-similar shape.

C is a fractal





all of C vs just the left side.

these sets are geometrically the same

The set is a rescaling of one of its parts.