Colloquium Thursday 10/27

Library basement, 1PM

(pizza 12:30)

Old Comp Question

a. Define what it means for a subset of $\mathbb{R}$ to be compact

K is compact means: If $(x_n) \in K$, then (x_n) has a convergent subsequence and the limit is in K.

b. Give an example of a bounded set that is not compact.

$K = \{1/n \mid n \in \mathbb{N}\}$

$(1/n) \in K$, $\frac{1}{n} \to 0$ but $0 \notin K$.

or $K = (0, 1)$

c Prove from the definition that $\{0\}$ is compact.

Pf Let $(x_n) \in \{0\}$

Then $(x_n) = (0, 0, 0, 0, \dots)$

This has a convergent subseq. (itself), it converges to 0, and $0 \in \{0\}$.

d Prove $\mathbb{N}$ is not compact

$(x_n) = (n) \in \mathbb{N}$,

has no conv. subseq.

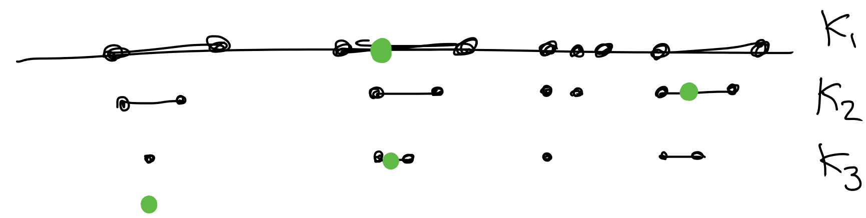
So $\mathbb{N}$ is not compact.

NIP for compact sets.

Thm Let K_n be a sequence of nested nonempty compact sets.

So $K_{n+1} \subseteq K_n$

Then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.



Pf Since $K_n \neq \emptyset$ for each n,

$\exists$ a sequence (x_n) with $x_n \in K_n$ for each n.

All $(x_n) \in K_1$ for every n,

and K_1 is compact, so x_n has a conv. subseq, ~~and~~ the limit is in K_1.

call it x.

all of x_n (except maybe x_1) are also in K_2.

So $(x_n) \to x$

these are in K_2 (eventually)

So $x \in K_2$ since K_2 is compact.

Similarly, $x \in K_3$ and $x \in K_4$, etc.

So $x \in K_n$ for all n.

So $x \in \bigcap_{n=1}^{\infty} K_n$, so $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$

Open covers & subcovers

Def Given $A \subseteq \mathbb{R}$, an open cover is a collection sets (typically intervals) which cover A.

i.e. $A \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ $\qquad \bigcup_{i=1}^{\infty} \mathcal{O}_i$

each $\mathcal{O}_\lambda$ is an open set.

A finite open cover is when the # of open sets is finite.

A subcover is a choice of only some of the open sets in an open cover.

Ex1 $A = [0, \infty)$

$$\mathcal{O}_n = (n-1, n+1) \qquad n \in \{0, 1, 2, \dots\}$$

we have $\mathcal{O}_0 = (-1, 1)$, $\mathcal{O}_1 = (0, 2)$, $\mathcal{O}_2 = (1, 3)$, ...

The $\mathcal{O}_n$ make an open cover of A.

This is not a finite cover,
and there is no finite subcover.
(taking only finitely many of these, it won't cover A)

$A = [0, 10]$ 

$\mathcal{O}_n = (n-1, n+1)$ also covers this A.

This has a finite subcover.

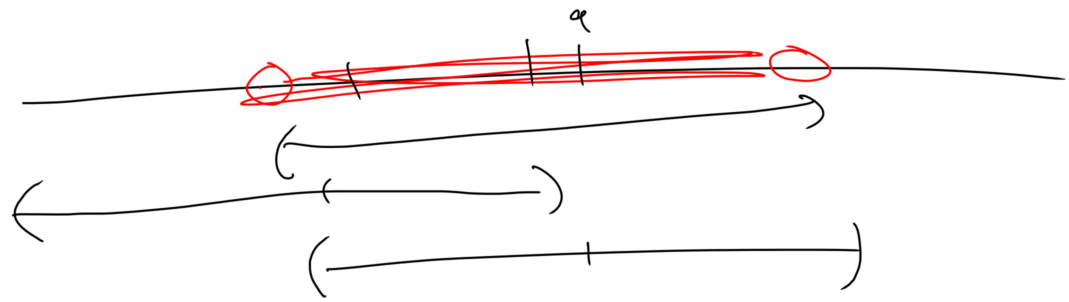
If A is unbounded, then some open covers will not have finite subcovers.

Contrapos: If all open covers have finite subcovers, then A is bounded.

Weirder examples:

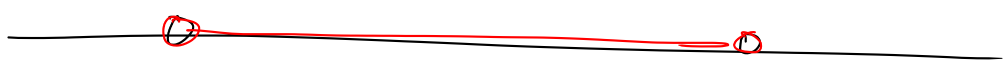
$A = (0, 1)$, $\mathcal{O}_q = (q - 1/2, q + 1/2)$

for every $q \in \mathbb{Q}$.



Does it have a finite subcover?

$\mathcal{O}_{1/2}$ covers all of A.



Covered by $\mathcal{O}_x = (0, x)$ for $x \in (0,1)$.