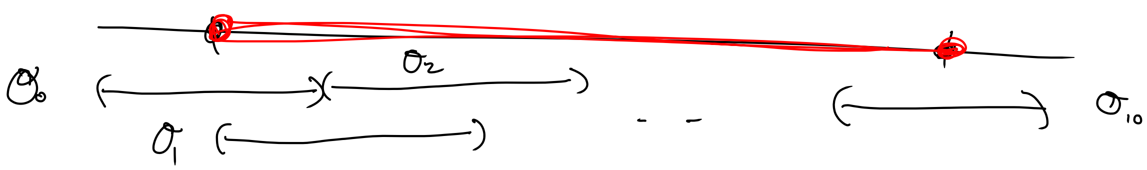
Open covers & finite subcovers

an open cover of $A \subseteq \mathbb{R}$ is a collection of open sets $\{\mathcal{O}_\lambda\}$ which cover A:

$$A \subseteq \bigcup_\lambda \mathcal{O}_\lambda$$

$A = [0, 10]$, $\qquad \mathcal{O}_n = (n-1, n+1)$

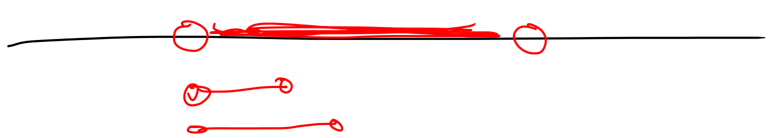


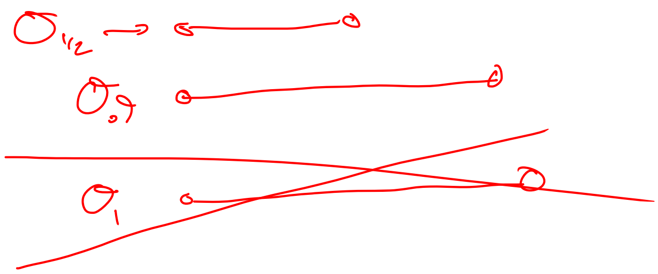
$\mathcal{O}_n$ for all $n \in \mathbb{Z}$ is an open cover.

There is a finite subcover $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_{10}$

Strange example:

$A = (0, 1) \qquad \mathcal{O}_x = (0, x)$ for $x \in (0,1)$

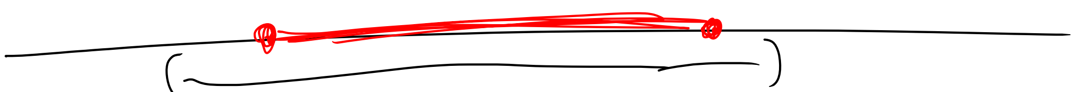




$\{\mathcal{O}_x\}$ for $x \in (0,1)$ is an open cover of $(0,1)$,

but it has no finite subcover.

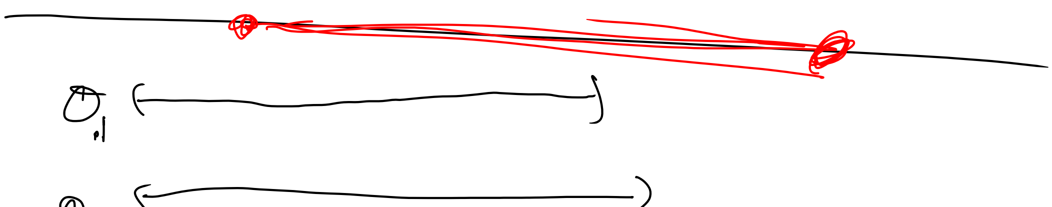
$A = [0,1]$



~~Let $\mathcal{O}_x = (0, x)$ for $x \in (0,1)$~~

~~This $\mathcal{O}_x$ is not an open cover.~~

$\mathcal{O}_x = (-1/2, x + 1/2)$ for $x \in (0,1)$



$\mathcal{O}_{.6} = (-1/2, 1.1)$

covers all of A.

Fact: For a closed interval, any open cover has a finite subcover.

The Heine-Borel Theorem

The following are equivalent:

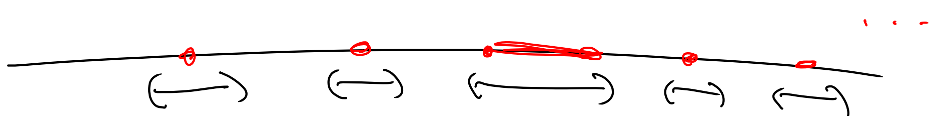
- A is compact ①
- A is closed & bounded ②
- Any open cover of A has a finite subcover. ③

We already saw $1 \Leftrightarrow 2$

We'll discuss $2 \Leftrightarrow 3$

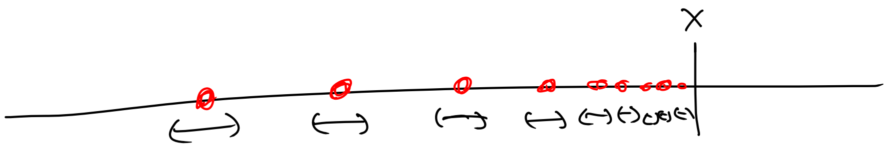
<u>$3 \Rightarrow 2$</u> Assume any open cover has a finite subcover. WTS A is closed & bounded.

<u>bounded</u> If A is unbounded, then cover A with small nbhds of every pt.



This has no finite subcover.

closed Let x be a limit pt of A,
WTS $x \in A$.



Make an open cover with small nbhds around these pts. FSOC, assume $x \notin A$, and make the nbhds small enough to miss x. Then there will be no finite subcover. $\rightarrow\leftarrow$.

$2 \Rightarrow 3$ is in the book.

Comp Q:

Prove using Heine-Borel that $\mathbb{Z}$ is not compact.

Pf $\mathbb{Z}$ is not bounded, so $\mathbb{Z}$ is not closed & bounded, so by HB, $\mathbb{Z}$ is not compact.

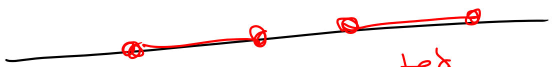
Connected Sets

A set is connected when it's all in one piece.

$[a,b]$ or (a,b)

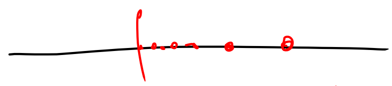
these are connected.

$[1,2] \cup [3,4]$



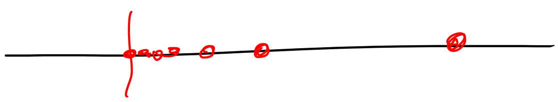
disconnected.

$\{1/n\}$



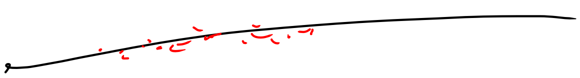
disconnected

$\{1/n\} \cup \{0\}$



disconnected

$\mathbb{Q}$



disconnected.

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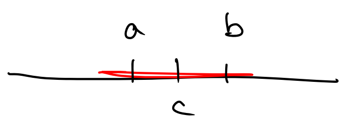
2 definitions:

Easy to understand, hard to use:

based on "between"

$E \subseteq \mathbb{R}$ is connected means: If $a < b$ and $a, b \in E$ and $a < c < b$, then $c \in E$.



Def E is <u>disconnected</u> when we can write $E = A \cup B$ such that

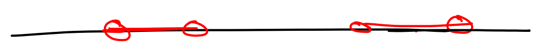
$$\bar{A} \cap B = \emptyset \quad \text{and} \quad A \cap \bar{B} = \emptyset.$$

each of A & B are called <u>components</u>

and "$E = A \cup B$" is a <u>separation</u> of E.

E is <u>connected</u> when there is no separation of E.

Ex] $(0,1) \cup (3,4)$



This is disconnected because we can make a separation

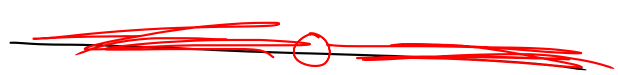
$A = (0,1)$

$B = (3,4)$

and $\bar{A} \cap B = [0,1] \cap (3,4) = \emptyset$

and $A \cap \bar{B} = (0,1) \cap [3,4] = \emptyset$

$E = \mathbb{R} - \{0\}$



it is disconnected:

use $A = (-\infty, 0)$

$B = (0, \infty)$

$\bar{A} \cap B = \overline{(-\infty, 0)} \cap (0, \infty) = (-\infty, 0] \cap (0, \infty) = \emptyset$

$A \cap \bar{B} = (-\infty, 0) \cap [0, \infty) = \emptyset$

$\mathbb{Q}$

$\mathbb{Q}$ is disconnected

$A = (-\infty, \pi) \cap \mathbb{Q}$

$B = (\pi, \infty) \cap \mathbb{Q}$

$\bar{A} \cap B = (-\infty, \pi] \cap (\pi, \infty) \cap \mathbb{Q} = \emptyset$

$A \cap \bar{B} = (-\infty, \pi) \cap \mathbb{Q} \cap [\pi, \infty) = \emptyset.$