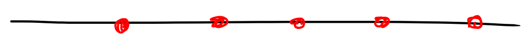
Def E is disconnected when you can write $E = A \cup B$ with

$$\bar{A} \cap B = \emptyset$$
and $$A \cap \bar{B} = \emptyset.$$

E is connected when it's not disconnected.

$\mathbb{N}$



$$\mathbb{N} = A \cup B$$
$$A = \{1, 3, 5, \dots\}$$
$$B = \{2, 4, 6, \dots\}$$

$$\bar{A} \cap B = \{1, 3, 5, \dots\} \cap \{2, 4, 6 \dots\} = \emptyset$$
$$A \cap \bar{B} = \{1, 3, 5, \dots\} \cap \{2, 4, 6, \dots\} = \emptyset.$$

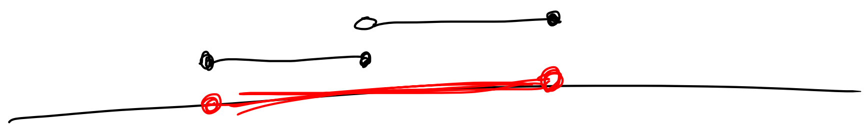
Could've done
$$A = \{1\}$$
$$B = \{2, 3, \dots\}$$

It's possible that $A \cap B = \emptyset$

but $\bar{A} \cap B \neq \emptyset$

We can do $E = A \cup B$ even for

$E = [0, 2]$



Could do $E = A \cup B$

with $A = [0, 1)$

$B = [1, 2]$

This is not a separation, since

$\bar{A} \cap B = [0,1] \cap [1,2] = \{1\} \neq \emptyset.$

<u>Thm</u> E is connected (no separation)

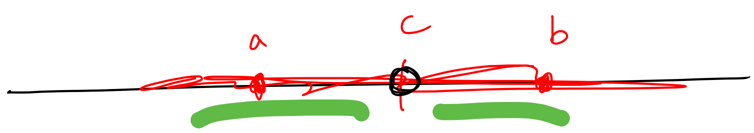
<u>iff</u> if $a, b \in E$ with $a < c < b$,

then $c \in E$.

$\Longrightarrow$ Assume E is connected, and

let $a, b \in E$ with $a < c < b$,

then we'll show $c \in E$.



FSOC assume $c \notin E$.
We'll show this implies a separation of E.

$$E = A \cup B$$

$$A = (-\infty, c) \cap E$$

$$B = (c, \infty) \cap E$$

Then $\bar{A} \cap B$: $\bar{A} = \overline{(-\infty, c) \cap E}$
$= (-\infty, c] \cap E$

$$\hat{A} \cap B = (-\infty, c] \cap E \cap (c, \infty) \cap E = \emptyset$$

$$A \cap \bar{B} = \text{— — —} \quad \emptyset$$

So E is disconnected $\times$.

Basically all connected sets of real #s
look like intervals (open or closed or half),

or single points.

Functions

Specifically: $\lim_{x \to 2} f(x)$ ← ε-type stuff.

Functions can be wild

pre 1800s – people thought of particular important functions.

polynomials, trig, log, exp, etc.

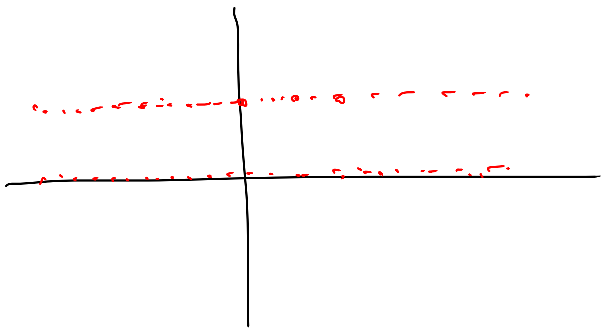
these all tend to be continuous & differentiable.

When we got set theory, we think of functions as any rule giving answers from some domain.

Dirichelet's Function

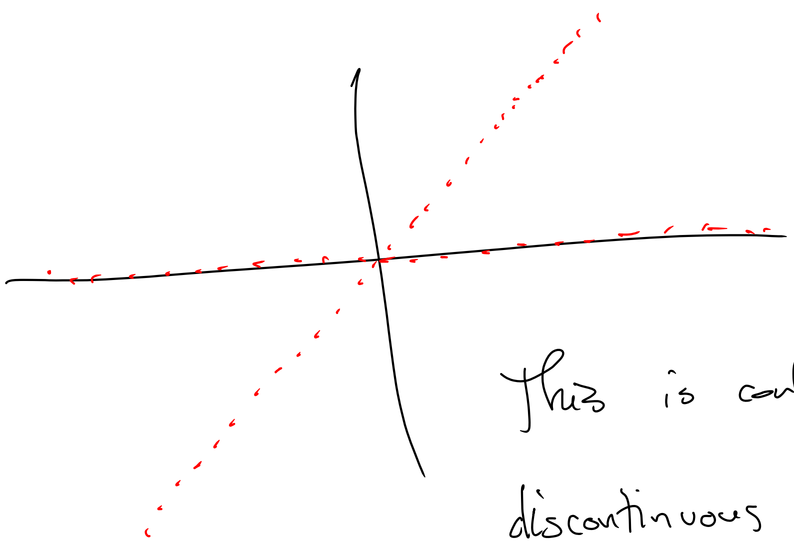
$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$



f is <u>nowhere</u> <u>continuous</u>

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$



This is continuous at $x=0$,
discontinuous everywhere else.

Thomae's Function

$$f(x) = \begin{cases} 1/n & \text{if } \quad x \in \mathbb{Q},\ x = \frac{m}{n} \\ 0 & \text{if } \quad x \notin \mathbb{Q} \end{cases}$$

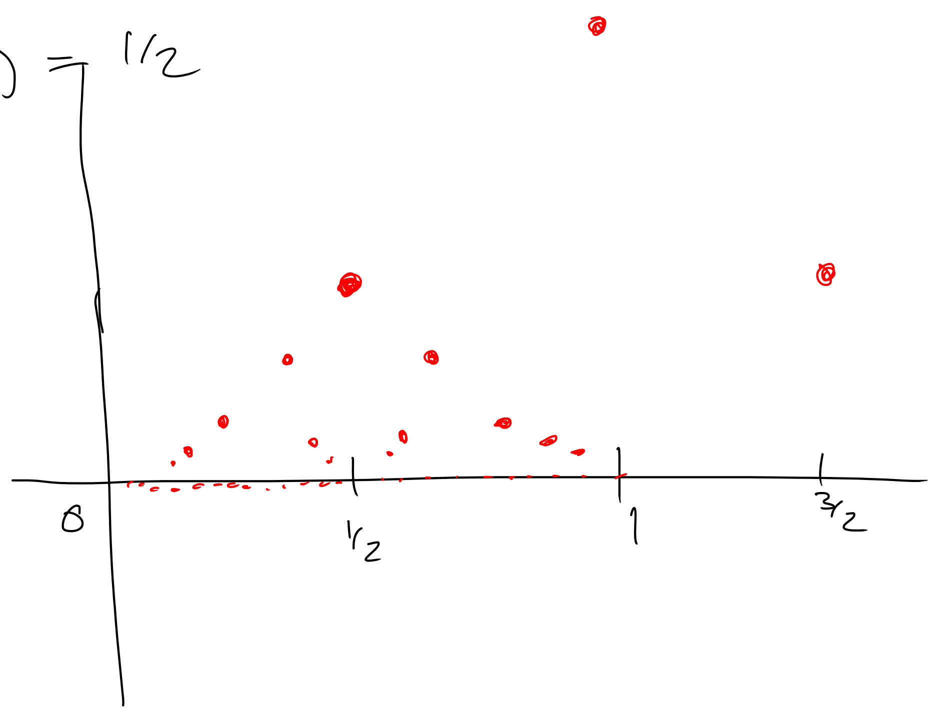
write x as a reduced fraction,

use only the denom.

$f(1) = 1$

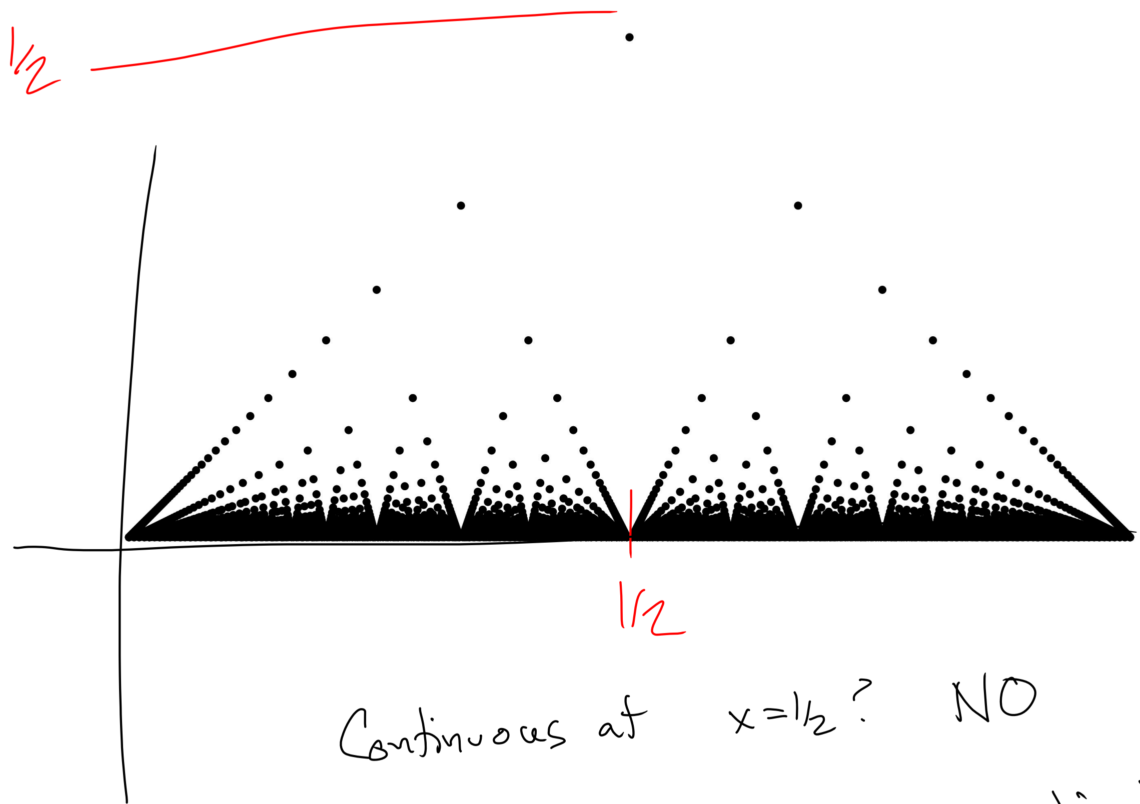
$f(1/2) = 1/2$

$f(3/2) = 1/2$



$f(1/3) = 1/3$ $\quad f(1/4) = 1/4$ $\quad f(3/4) = 1/4$

$f(2/3) = 1/3$ $\quad f(2/4) = 1/2$



Continuous at $x = 1/2$? NO

$f(x)$ is discontinuous at every rational.

$f(x)$ <u>is</u> continuous at every irrational.

i.e. $f(\sqrt{2}) = 0$ and when x is really close to $\sqrt{2}$, we get $f(x)$ really close to 0.

$\sqrt{2} = 1.41421356\ldots$

a rational # really close to $\sqrt{2}$:

$x = 1.41421356$

is $f(x) \approx 0$? $\uparrow$

$$x = \frac{141421356}{100000000}$$

$$= \frac{2828427}{2000000}$$

So $f(x) = \frac{1}{2000000} \approx 0$.