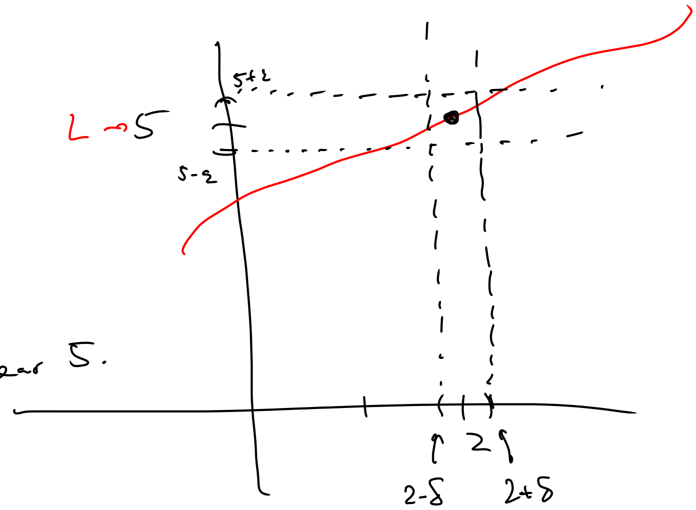
Limits of Function

$$\lim_{x \to 2} f(x) = 5$$



If x values are near 2,
then y-values are near 5.

it means: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
if $0 < |x-2| < \delta$, then $|f(x) - 5| < \varepsilon$

Def $\lim_{x \to c} f(x) = L$ means:

$$\boxed{\forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } \underbrace{0 < |x-c| < \delta}_{\text{if } x \text{ is close to } c} \ \underset{\text{then}}{\Rightarrow} \ \underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is close to } L}.}$$

In terms of neighborhoods: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$x \in V_\delta(c)$ and $x \neq c$ $\Rightarrow$ $f(x) \in V_\varepsilon(L)$

Ex1 Show $\lim_{x \to 4} 2x = 8$

Pf Let $\varepsilon > 0$ be given, We'll find $\delta > 0$ s.t.

$$0 < |x-4| < \delta \Rightarrow |2x-8| < \varepsilon.$$

"solve for $|x-4|$"

$|2x-8| = 2|x-4|$ want $2|x-4| < \varepsilon$,

so $|x-4| < \varepsilon/2$

when simplifying, use $=$ or $\leq$ or $<$

Let $\delta = \varepsilon/2$. Then if $|x-4| < \delta$, we have:

$$|2x-8| = 2|x-4| < 2\delta = 2 \cdot \varepsilon/2 = \varepsilon$$

as desired.

Show $\lim_{x \to 5} 2x+3 = 13$

Pf Let $\varepsilon > 0$ be given. We'll find $\delta > 0$ s.t.

$$0 < |x-5| < \delta \Rightarrow |2x+3-13| < \varepsilon$$

$|2x+3-13| = |2x-10| = 2|x-5|$

want $2|x-5| < \varepsilon$

$|x-5| < \varepsilon/2$

Let $\delta = \varepsilon/2$. Then if $|x-5| < \delta$, we have

$$|2x+3-13| = 2|x-5| < 2\cdot \varepsilon/2 = \varepsilon$$

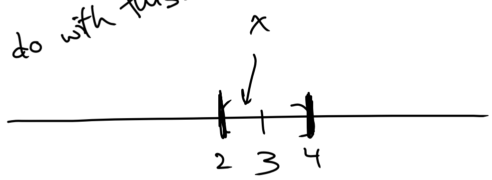
Shown.

$$\lim_{x\to 3} x^2 - x = 6$$

Pf Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$$0<|x-3|<\delta \Rightarrow |x^2-x-6| < \varepsilon$$

$$|x^2-x-6| = \underbrace{|x-3|}_{\delta}\,\underbrace{|x+2|}_{\text{what to do with this?}}$$



Assume $\delta < 1$,

then $x < 4$, so $x+2 < 6$

$< |x-3|\cdot 6$ want $|x-3|\cdot 6 < \varepsilon$

$|x-3| < \varepsilon/6$

Let $\delta < \min(1, \varepsilon/6)$. Then let $|x-3|<\delta$, we have:

$$|x^2-x-6| = |x-3|\,|x+2| < \frac{\varepsilon}{6}\cdot 6 = \varepsilon$$

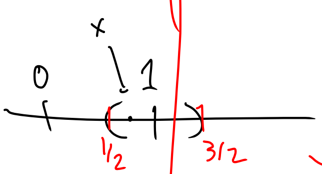
$$\lim_{x\to 1} \frac{5+x}{x} = 6$$

Pf. Let $\varepsilon>0$ be given, we'll find $\delta>0$ s.t.

$$0<|x-1|<\delta \Rightarrow \left|\frac{5+x}{x}-6\right|<\varepsilon.$$

$$\left|\frac{5+x}{x}-6\right| = \left|\frac{5+x}{x}-\frac{6x}{x}\right| = \left|\frac{5-5x}{x}\right|$$

$$= 5\left|\frac{1-x}{x}\right| = 5\cdot\frac{|1-x|}{|x|} = 5\cdot|x-1|\cdot\frac{1}{|x|}$$



Let $\delta<1/2$, then $x>1/2$.

Then $\leq 5|x-1|\cdot\frac{1}{1/2} = 10|x-1|$

So $|x-1|<\varepsilon/10$

Let $\delta < \min(1/2, \varepsilon/10)$. Then if $|x-1|<\delta$, we have:

$$\left|\frac{5+x}{x}-6\right| = 5|x-1|\cdot\frac{1}{|x|} \leq 10|x-1|$$

$$< 10\cdot\varepsilon/10 = \varepsilon$$

Shazam!