$$\lim_{x \to c} f(x) = L \quad \text{means:}$$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$\underbrace{0 < |x-c| < \delta}_{\text{when } x \text{ is close to } c} \Rightarrow \underbrace{|f(x) - L| < \varepsilon}_{f(x) \text{ is close to } L}$$

---

$$\lim_{x \to 3} \frac{3}{x} = 1$$

Pf  Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$$0 < |x-3| < \delta \Rightarrow \left| \frac{3}{x} - 1 \right| < \varepsilon .$$

solve for $|x-3|$

$$\left| \frac{3}{x} - 1 \right| = \left| \frac{3}{x} - \frac{x}{x} \right| = \left| \frac{3-x}{x} \right| = \frac{|3-x|}{|x|}$$

$$= |3-x| \cdot \frac{1}{|x|} = |x-3| \cdot \frac{1}{|x|}$$

If $\delta < 1$, then $|x-3| < 1$ so $2 < x < 4$

so $\frac{1}{|x|} \leq \frac{1}{2}$

$$\leq |x-3| \cdot \frac{1}{2}$$

So we want $|x-3| \cdot \frac{1}{2} < \varepsilon$

$|x-3| < 2\varepsilon$

Let $\delta < 2\varepsilon$.

Let $\delta < \min(1, 2\varepsilon)$. Then if $|x-3|<\delta$, we have

$$\left| \frac{3}{x} - 1 \right| = |x-3| \cdot \frac{1}{|x|} \le 2\varepsilon \cdot \frac{1}{2} = \varepsilon \text{ as desired.}$$

---

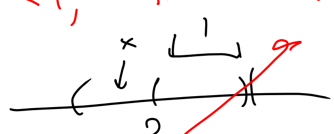
$$\lim_{x \to 2} x^2 = 4$$

Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$$0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$$

$$|x^2-4| = |(x+2)(x-2)| = \boxed{|x-2| \cdot |x+2|} \le |x-2| \cdot$$

let $\delta < 1$, then $|x-2| < 1$




$1 < x < 3$

so $3 < \boxed{x+2 < 5}$

$\le |x-2| \cdot 5$

$|x-2| < 1$

Same as $-1 < x-2 < 1$

$1 < x < 3$

Let $\delta < \min(1, \varepsilon/5)$. Then if $|x-2| < \delta$, we have:

$$|x^2-4| = |x-2||x+2| \le \varepsilon/5 \cdot 5 = \varepsilon$$

Shown.

$$\lim_{x \to 2} x^3 = 8$$

Pf let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t

$0 < |x-2| < \delta \Rightarrow |x^3 - 8| < \varepsilon$

$a^3 - b^3$

$= (a-b)(a^2 + ab + b^2)$

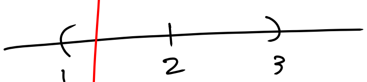
$|a+b| \leq |a| + |b|$

$|x^3 - 8|$

$= |x-2|\,|x^2 + 2x + 4|$

good

let $\delta < 1$, so $1 < x < 3$




$$|x-2|\,|x^2 + 2x + 4| \leq |x-2|\left(|x^2| + |2x| + |4|\right)$$
$$= |x-2|\left(|x|^2 + 2|x| + 4\right)$$
$$\leq |x-2|\left(3^2 + 2\cdot 3 + 4\right) = |x-2|\cdot 19$$

Let $\boxed{\delta < \min(1, \varepsilon/19)}$

then if $|x-2| < \delta$, we have:

$$|x^3 - 8| = |x-2|\,|x^2 + 2x + 4| \leq |x-2|\cdot 19 < \varepsilon/19 \cdot 19 = \varepsilon$$

Shown.

$$\lim_{x\to 2} x^{10} = 2^{10}$$

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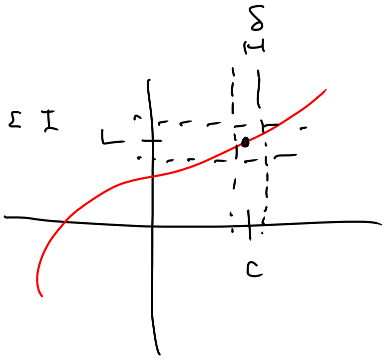
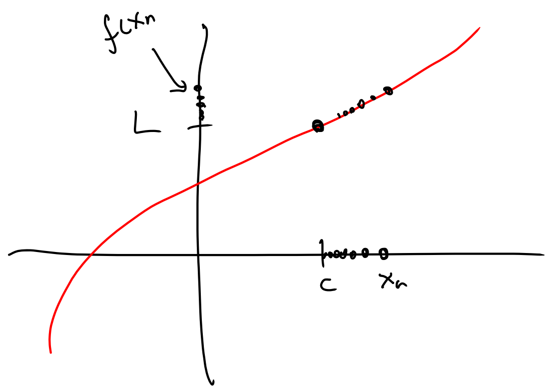
# A Sequential Criterion for limits

$$\lim_{x\to c} f(x) = L \iff \text{For all sequences } x_n \to c, \text{ we have } f(x_n) \to L$$




Use the sequence version to show some limits do not exist.