Sequential Criterion for Limits of functions

$\lim_{x \to c} f(x) = L \iff$ for all seqs $(x_n) \to c$, we have $f(x_n) \to L$.

We can rephrase limits of functions as lims of sequences.

For free, we get:

Thm If $\lim_{x\to c} f(x) = L$ & $\lim_{x \to c} g(x) = M$ & $k \in \mathbb{R}$, then

i) $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x) = kL$

ii) $\lim_{x\to c}\big(f(x) + g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$

iii) $\lim_{x\to c}\big(f(x)\cdot g(x)\big) = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right) = L \cdot M$

iv) $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$ when $M \neq 0$.

How to show limits do not exist?

$\lim_{x\to 3} \frac{1}{x-3}$ does not exist.

i.e. there is no L with

$\lim_{x\to 3} \frac{1}{x-3} = L$.

Hard to demonstrate using $\varepsilon - \delta$ definition.

But not so bad with sequences.

$\lim_{x\to 3} \frac{1}{x-3} = L \iff \forall$ sequences $x_n \to 3$,

we have $\frac{1}{x_n-3} \to L$.

not so hard to negate:

$\lim_{x\to 3} \frac{1}{x-3} \neq L \iff \exists$ a seq. $x_n \to 3$

s.t. $\frac{1}{x_n-3} \not\to L$.

We need a sequence $x_n \to 3$,

but $\frac{1}{x_n-3}$ does not approach any L.

Try $x_n = 3 + 1/n$ $\quad x_n \to 3$, and:

$$\frac{1}{x_n-3} = \frac{1}{3+\frac{1}{n}-3} = \frac{1}{1/n} = n$$

so $\left(\frac{1}{x_n-3}\right) = (n)$ which does not approach any limit.

so $\lim\limits_{x\to 3} \frac{1}{x-3}$ does not exist

Thm $\lim\limits_{x\to c} f(x)$ does not exist

$\iff \exists\, (x_n) \to c$ with $(f(x_n))$ divergent.

Ex $\lim\limits_{x\to 0} \sin(1/x)$ does not exist

Pf ~~Let $x_n = (1/n)$, then~~

$$\cancel{\sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{1/n}\right) = \sin n}$$

$$\cancel{x_n = \left(\frac{1}{\pi n}\right)}$$

$$\cancel{\sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{1/\pi n}\right) = \sin \pi n = 0}$$

Let $x_n = \left(\frac{1}{\frac{\pi}{2} n}\right)$

$$\sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{1/\frac{\pi}{2} n}\right) = \sin \frac{\pi}{2} n$$

$(1, 0, -1, 0, 1, 0, -1, 0, \ldots)$

diverges since we have a subseq conv. to 0, and another conv. to 1.

$\underline{\text{Thm}}$ $\lim_{x \to c} f(x)$ does not exist if

$\exists\, x_n \to c$ and $y_n \to c$

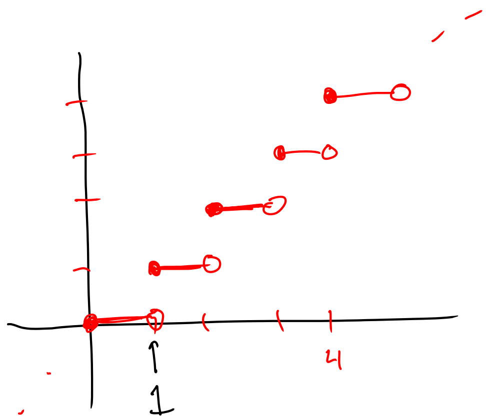
with $\lim f(x_n) \neq \lim f(y_n)$

$\underline{\text{Ex}}$ $\lim_{x \to 1} [[x]]$ ← greatest integer less than or equal to x.

$[[4]] = 4$

$[[3.7]] = 3$

$[[\pi]] = 3$



$\lim_{x \to 1} [[x]]$ does not exist.

use $x_n = 1 + 1/n$ show $\lim f(x_n) \neq \lim f(y_n)$

$y_n = 1 - 1/n$

$f(x_n) = [[1 + 1/n]] = (2, 1, 1, 1, 1, \ldots)$

$f(y_n) = [[1 - 1/n]] = (0, 0, 0, 0, 0, \ldots)$

So $\lim f(x_n) = 1$ not the same!

$\lim f(y_n) = 0$

So $\lim_{x \to 1} [[x]]$ does not exist.

To show lims do not exist, use sequences

To show lims exist, you must use ε - δ proofs. Demonstrating a sequence is not enough.

Ex $\lim_{x \to 0} \sin \frac{1}{x}$: choose $x_n = \frac{1}{\pi n}$

then $f(x_n) = \sin \pi n = (0, 0, 0, 0 \ldots)$
converges!

The seq. may converge, even if the function limit does not exist.

Continuity

Def A function $f(x)$ is continuous at c when $\lim_{x \to c} f(x) = f(c)$

in detail: $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

with sequences: for all seqs $x_n \to c$, we have $f(x_n) \to f(c)$

Ex $f(x) = \sqrt{x}$ is continuous at $x=4$.

Pf $\lim_{x\to 4} \sqrt{x} = \sqrt{4} = 2$ WB $\lim_{x\to 4} \sqrt{x} = 2$

Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$0 < |x-4| < \delta \Rightarrow |\sqrt{x} - 2| < \varepsilon$.

$$|\sqrt{x}-2| = \left|(\sqrt{x}-2)\frac{\sqrt{x}+2}{\sqrt{x}+2}\right| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right|$$

$$= \left|\frac{x-4}{\sqrt{x}+2}\right| = |x-4| \cdot \frac{1}{|\sqrt{x}+2|} \le |x-4| \cdot \frac{1}{\sqrt{3}+2}$$

$$\le |x-4|$$

Let $\delta < 1$, so $3 < x < 5$

Let $\delta = \min(1, \varepsilon)$

etc.