Continuity

Def $f(x)$ is <u>continuous</u> <u>at</u> <u>c</u> when:

$$\lim_{x\to c} f(x) = f(c).$$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

Ex1 $f(x) = x^2$ is continuous at 3

Pf WTS $\lim_{x\to 3} x^2 = 3^2$ $\qquad \lim_{x\to 3} x^2 = 9$

prove this with ε-δ's.

To show $f(x)$ is not continuous at c,

must show $\lim_{x\to c} f(x) \neq f(c)$

probably best to find a sequence

$$x_n \to c$$

but $f(x_n) \not\to f(c)$

Some subtlety:

In calc, we say $f(x) = \frac{1}{x}$ is discontinuous at $x = 0$.

We don't say this, because $x=0$ is outside the domain of f.

$\frac{1}{x}$ has domain $\mathbb{R} - \{0\}$

we don't ask "is $f(x) = 1/x$ continuous at 0?"

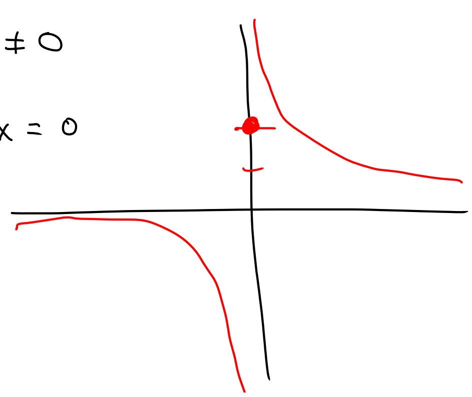
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
$0 < |x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$

there is no $f(0)$,
so this doesn't make sense.

We'll just say 0 is outside the domain.

We could consider

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$



This is not continuous at $x=0$.

i.e. $\lim_{x\to 0} f(x) \neq f(0)$

i.e $\lim_{x\to 0} f(x) \neq 2$

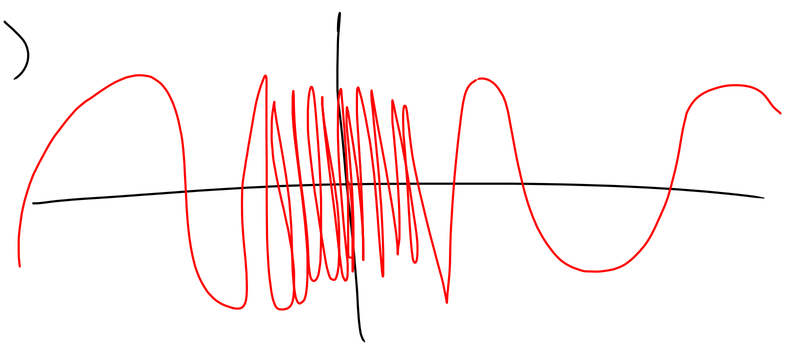
WTS a sequence $x_n \to 0$, but $f(x_n) \not\to 2$.

try $x_n = \frac{1}{n}$ so $x_n \to 0$

then $f(x_n) = f(1/n) = \frac{1}{1/n} = n$

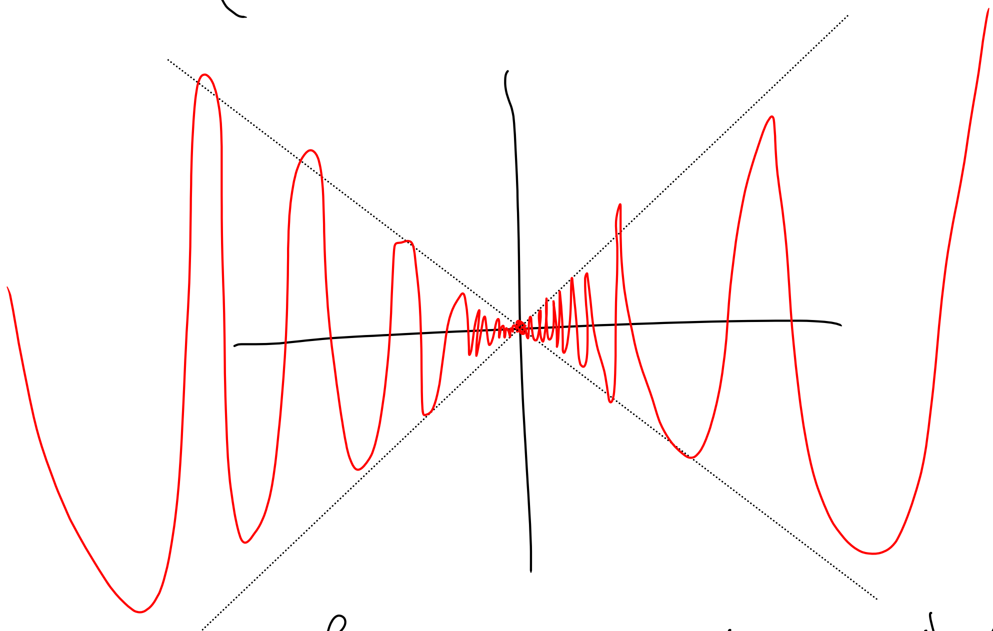
So $(f(x_n)) = n$ is unbounded, so diverges, so $f(x_n) \not\to 2$.

$\sin(1/x)$



$x=0$ is outside the domain.

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



This is continuous at 0.

Pf WTS $\lim_{x\to 0} f(x) = 0$

i.e $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \varepsilon$$

$$0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon.$$

(since $0<|x|$)

$$|f(x)| = |x\sin(\tfrac{1}{x})| = |x|\,|\sin(\tfrac{1}{x})| \leq |x| \cdot 1 = |x|$$

need $|x| < \varepsilon$,

let $\delta = \varepsilon$, then $0<|x|<\delta \Rightarrow$

$$|f(x)| = |x \sin(1/x)| \leq |x| < \delta = \varepsilon$$

so $|f(x)| < \varepsilon$ as desired.

Old comp question (obscure)

Prove from the definition that if f & g are continuous at $a \in \mathbb{R}$, then $f+g$ is continuous at a.

Pf Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$|a+b| \leq |a|+|b|$ $\quad 0 < |x-a| < \delta \Rightarrow \left| f(x)+g(x) - (f(a)+g(a)) \right| < \varepsilon$

$$|f(x)+g(x) - f(a) - g(a)| = |f(x) - f(a) + g(x) - g(a)|$$

$$\leq |f(x) - f(a)| + |g(x) - g(a)|$$

Choose δ so small that $0 < |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon/2$ and $|g(x)-g(a)| < \varepsilon/2$

Then $|f(x)+g(x) - f(a) - g(a)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$

Shown.

Thm If f & g are continuous at c, then:

i) $kf(x)$ is continuous at c

ii) $f(x)+g(x)$ is continuous at c

iii) $f(x)\cdot g(x)$ is continuous at c

iv) $f(x)/g(x)$ is continuous at c, as long as $g(c) \neq 0$.

Cor

- any polynomial is continuous at any $c \in \mathbb{R}$
- any ratio of polynomials is continuous at all points in its domain.

Next Time:

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

is not continuous at any $c \in \mathbb{Q}$,
continuous for all $c \notin \mathbb{Q}$.