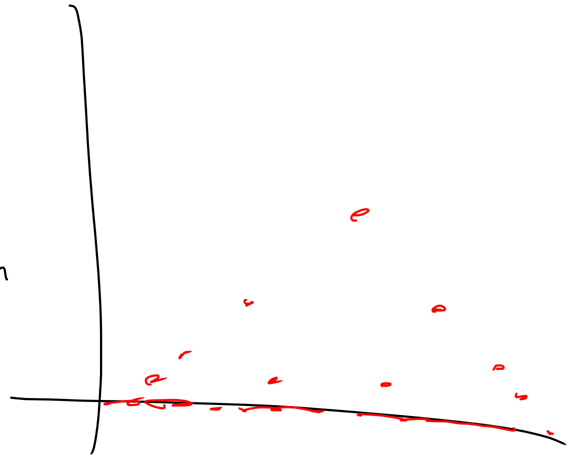
Thomae's function

$$f(x) = \begin{cases} 1/n & \text{if } x \in \mathbb{Q},\ x = m/n \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$



We'll show $f$ is discontinuous at all rationals,
continuous at all irrationals.

Let $c \in \mathbb{Q}$, WTS $f$ is discontinuous at $c$.

use sequences: find a seq $x_n \to c$
but $f(x_n) \not\to f(c)$

Say $c = m/n$, so $f(c) = 1/n$.
choose a seq of irrationals $x_n \to c$.
maybe $x_n = \boxed{c + \frac{\sqrt{2}}{n}}$

Then $f(x_n) = 0$, so $f(x_n) \to 0 \neq \underbrace{f(c)}_{1/n}$

Continuous at all irrationals: Let $c \notin \mathbb{Q}$, $f$ is cont. at $c$

Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

since $c \notin \mathbb{Q}$, this means

$|f(x)| < \varepsilon$.

Since $c \notin \mathbb{Q}$, all rationals very near $c$ have large denominator.

if $c = \pi \approx 3.1415...$

3.1415 is rational, near $\pi$,

$\hookrightarrow \frac{31415}{10000} \rightarrow$ reduced, still has big denom.

Choose $\delta$ so that all rationals within $\delta$ of $c$ have denom $n > 1/\varepsilon$

Then if $|x-c|<\delta$, then

$|f(x) - f(c)| = |f(x)|$

if $x$ is irrational, $\rightarrow = 0 < \varepsilon$ as desired

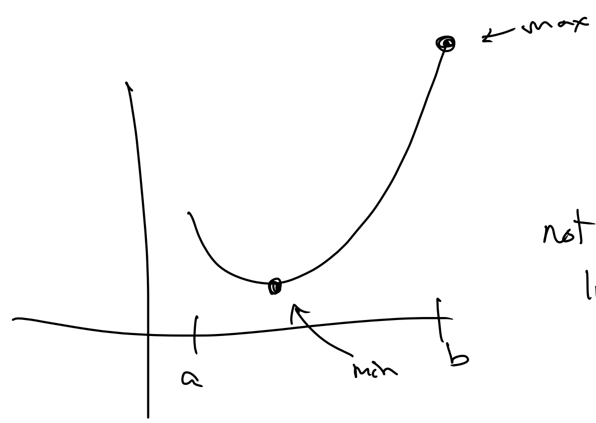
if $x$ is rational $\rightarrow \frac{1}{n} < \frac{1}{1/\varepsilon} = \varepsilon$ as desired.

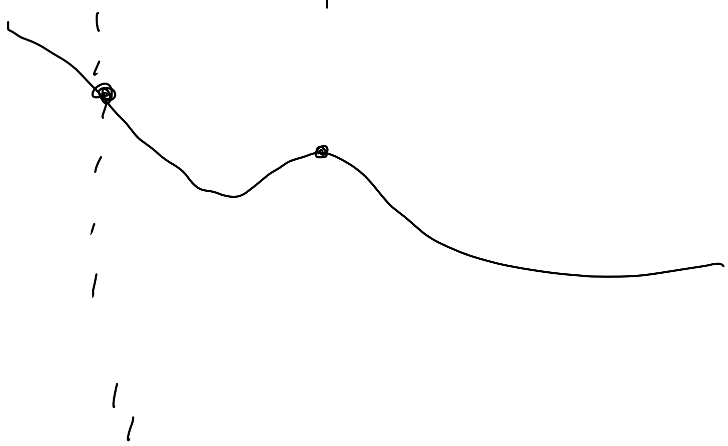
A classic calculus theorem!

The Extreme Value Theorem

calc: If $f$ is cont. on $[a,b]$, then $f$ has an abs max & min on $[a,b]$.

not true of open intervals, not true of closed sets like $[a,\infty)$



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<u>Thm</u> If $K \subseteq \mathbb{R}$ is compact and $f$ is continuous on $K$, then $f$ has a min & max value.

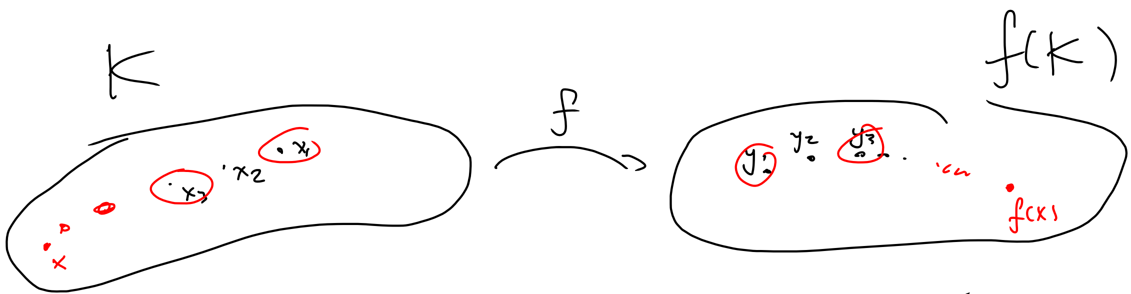
Better theorem:

<u>Thm</u> If $K$ is compact & $f$ is continuous on $K$, then $f(K)$ is compact.

Pf  WTS $f(K)$ is compact.

i.e. if $y_n \in f(K)$, then $\exists$ a conv. subseq converging to a pt of $f(K)$.

since $y_n \in f(K)$, $\exists$ a seq $x_n \in K$ s.t. $f(x_n) = y_n$ for each $n$.




Since $K$ is compact, $\exists$ a subseq $(x_{n_k})$ of $(x_n)$ s.t. $x_{n_k} \to x$ and $x \in K$.

and $y_{n_k} = f(x_{n_k}) \longrightarrow f(x)$ since $f$ is continuous

So $y_n$ has a subseq $y_{n_k}$, and $y_{n_k} \to f(x) \in f(K)$.

Use this to show $f$ has a min & max:

$f(K)$ is compact, so it's closed & bounded.

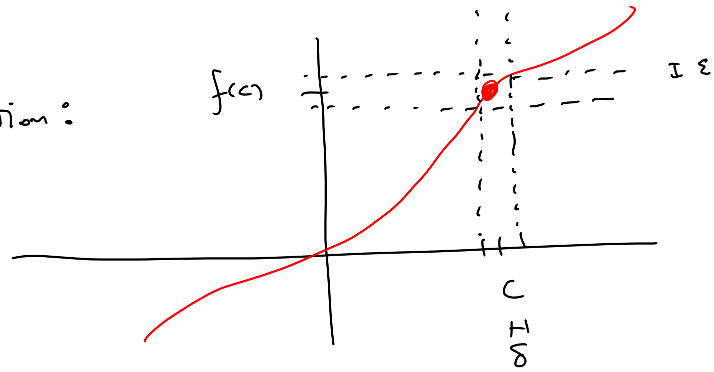
Since it's bounded, it has a sup.
Since it's closed, the sup is in the set,

So it's a max.

Similarly, min exists because inf in is the set.

# Uniform Continuity.

For a continuous function:




Typically, the $\delta$ depends on the $c$.

$f(x) = x^2$ is cont at $c = 3$:

$$|x^2 - 9| = |x-3||x+3|$$
$$\leq |x-3| \cdot 7$$

if $\delta < 1$, $x < 4$

so $x + 3 < 7$

Let $\delta = \min(1, \varepsilon/7)$