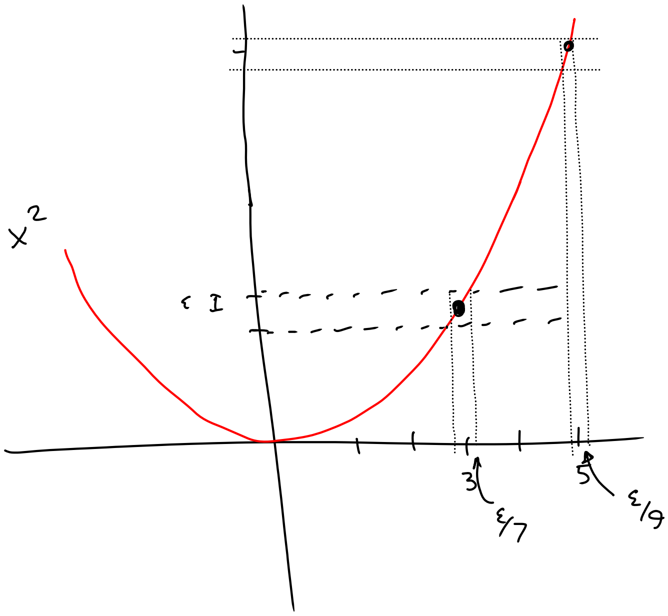
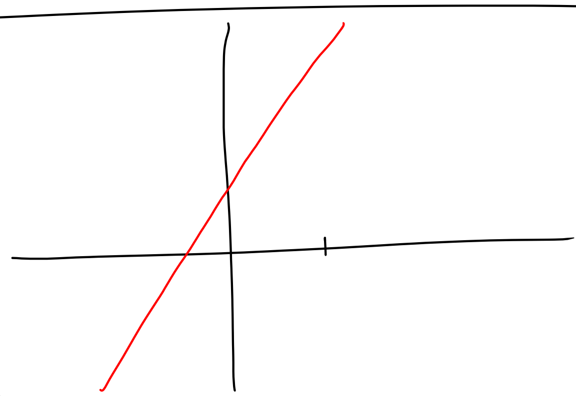
$f(x) = x^2$



$f(x)$ is continuous at $c = 3$ (use $\delta = \varepsilon/7$)

Typically, we need different δ as we consider different points c.

$f(x) = 3x+1$



For any point c,

$$\underbrace{|3x+1-(3c+1)|}_{|f(x)-f(c)|} = |3x+1-3c-1| = 3|x-c|$$

use $\delta = \varepsilon/3$

"Uniformly Continuous" means same δ works no matter what ϵ is

Def f is <u>uniformly continuous</u> on $A \subseteq \mathbb{R}$ when:

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $\forall x, y \in A$,

if $0 < |x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Ex) $f(x) = 5x - 10$ is unif. continuous on $\mathbb{R}$.

Pf let $\varepsilon > 0$ be given, we'll find δ s.t. $\forall x, y \in \mathbb{R}$,

$$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

$|f(x) - f(y)| = |5x - 10 - (5y - 10)| = |5x - 10 - 5y + 10|$

$= |5x - 5y| = 5|x-y|$

want $5|x-y| < \varepsilon$, so $|x-y| < \varepsilon/5$

let $\delta = \varepsilon/5$, then if $|x-y| < \delta$, we have

$|f(x) - f(y)| = 5|x-y| < 5 \cdot \varepsilon/5 = \varepsilon$ as desired.

Uniformly continous on A.

"ordinary continuity" is called <u>pointwise</u> <u>continuity</u>

"Not pointwise continuous"

find $x_n \to c$ but $f(x_n) \not\to f(c)$

"Not unif continuous on A"

<u>Thm</u> f is not unif. continuous on A when we can find sequences $x_n, y_n \in A$

$$|x_n - y_n| \to 0$$

but $|f(x_n) - f(y_n)| \not\to 0$.

<u>Ex</u> Show $f(x) = x^2$ is not unif continuous on $\mathbb{R}$.

Choose $x_n = n$ ~~then~~ $\cancel{|x_n - y_n| \to 0}$

$\cancel{y_n = n}$

~~and~~ $\cancel{|f(x_n) - f(y_n)| = |f(n) - f(n)| = 0}$

$x_n = n$ then $|x_n - y_n| = |n - (n + 1/n)|$

$y_n = n + 1/n$

$= |-1/n| = 1/n \to 0$ ✓

And $|f(x_n) - f(y_n)| = |f(n) - f(n + 1/n)|$

$= |n^2 - (n + 1/n)^2| = |n^2 - (n^2 + 2 + 1/n^2)|$

$= |-2 + 1/n^2| = 2$ so $|f(x_n) - f(y_n)| \not\to 0$. ✓

But $f(x) = x^2$ is unif continuous on $[4,8]$

Pf Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t. $\forall x, y \in [4,8]$,

$$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Since $x, y \in [4,8]$

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y||x+y|$$

$$\leq |x-y| \cdot (|x| + |y|) \leq |x-y| \cdot (8+8)$$

$$\leq |x-y| \cdot 16$$

Let $\delta = \varepsilon/16$. Then when $|x-y| < \delta$, we have

$$|f(x) - f(y)| = |x-y||x+y| \leq |x-y| \cdot 16$$

$$< \varepsilon/16 \cdot 16 = \varepsilon$$