f is unif. cont. on A means:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\forall x, y \in A$,

$$0 < |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

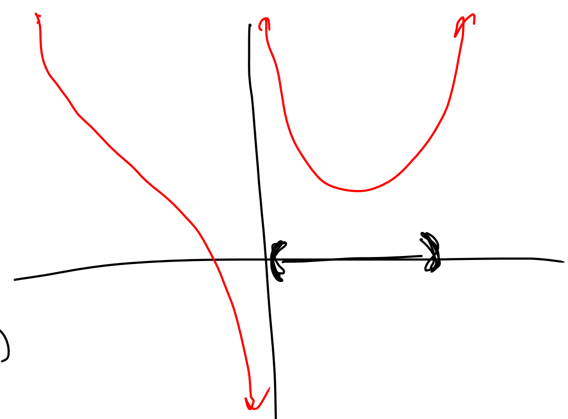
f is not unif cont on A if we can find seqs

$x_n, y_n \in A$ with

$|x_n - y_n| \to 0$

but $|f(x_n) - f(y_n)| \not\to 0$.

<u>Ex1</u> $f(x) = x^2 + \frac{1}{x}$



Not unif. continuous on $(0, 2)$

Choose x_n, y_n both $\to 0$ since near 0 is where the weirdness happens.

let $x_n = 1/n$ for y_n, try: $\frac{2}{n}$, $\frac{1}{n^2}$, $\frac{1}{n+1}$

$y_n = \frac{1}{n+1}$

then $|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| \longrightarrow 0$

and $\left| f(x_n) - f(y_n) \right| = \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right) \right|$

$$= \left| \left(\frac{1}{n}\right)^2 + \frac{1}{1/n} - \left(\left(\frac{1}{n+1}\right)^2 + \frac{1}{1/(n+1)} \right) \right|$$

$$= \left| \frac{1}{n^2} + \cancel{n} - \frac{1}{(n+1)^2} - \cancel{n} - 1 \right|$$

$$= \left| \cancelto{0}{\frac{1}{n^2}} - \cancelto{0}{\frac{1}{(n+1)^2}} - 1 \right| \longrightarrow \left| -1 \right| = 1 \neq 0$$

so $\left| f(x_n) - f(y_n) \right| \not\to 0$.

It is unif. cont on $[2,5]$

Pf Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t. $\forall x, y \in [2,5]$,

$$0 < |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

$$|f(x) - f(y)| = \left| x^2 + \frac{1}{x} - \left(y^2 + \frac{1}{y}\right) \right| = \left| x^2 + \frac{1}{x} - y^2 - \frac{1}{y} \right|$$

$$= \left| x^2 - y^2 + \frac{1}{x} - \frac{1}{y} \right| \leq \left| x^2 - y^2 \right| + \left| \frac{1}{x} - \frac{1}{y} \right|$$

$$= \left| (x-y)(x+y) \right| + \left| \frac{y-x}{xy} \right|$$

$$= |x-y||x+y| + |x-y| \cdot \frac{1}{|xy|}$$

$$= |x-y| \left(|x+y| + \frac{1}{|xy|} \right)$$

$\leq |x-y| \left((5+5) + \frac{1}{12 \cdot 21} \right)$

$\leq |x-y| \cdot 11$

Let $\delta = \varepsilon/11$, then if $|x-y| < \delta$, we have:

$|f(x) - f(y)| = \left| x^2 + \frac{1}{x} - (y^2 + \frac{1}{y}) \right| \leq |x-y| \cdot 11$
$< \varepsilon/11 \cdot 11 = \varepsilon$

Shown

$f(x) = \frac{1}{x-3}$

Find a set on which it isn't unif. cont. & prove it.

Find a set on which it is unif. cont & prove it.

Is unif cts: $[5,7]$

$|f(x) - f(y)| = \left| \frac{1}{x-3} - \frac{1}{y-3} \right| = \left| \frac{y-3-(x-3)}{(x-3)(y-3)} \right|$

$= |x-y| \cdot \left| \frac{1}{(x-3)(y-3)} \right|$

$\leq |x-y| \cdot \frac{1}{(5-3)(5-3)} = |x-y| \cdot \frac{1}{4}$

$\leq |x-y|$

Let $\delta = \cancel{4}\varepsilon$. etc.

Not unif cont. on $(3,5)$

let $x_n = 3 + 1/n$

$y_n = 3 + 2/n$

$$|x_n - y_n| = \left|3 + \tfrac{1}{n} - \left(3 + \tfrac{2}{n}\right)\right| = \left|\tfrac{1}{n} - \tfrac{2}{n}\right| = \left|-\tfrac{1}{n}\right| \to 0$$

$$|f(x_n) - f(y_n)| = \left|\frac{1}{3+\frac{1}{n}-3} - \frac{1}{3+\frac{2}{n}-3}\right| = \left|n - \frac{n}{2}\right|$$

$$= \left|n/2\right| \not\to 0$$

unbounded.