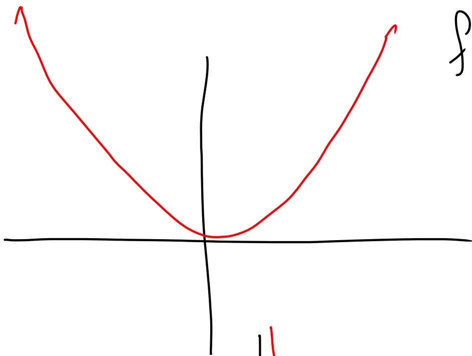
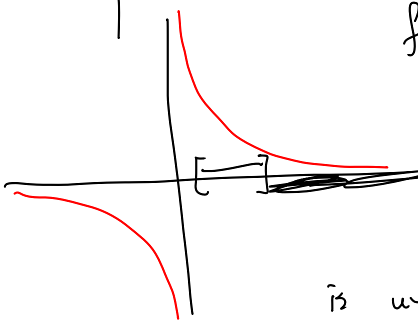
Unif. continuity



$f(x)=x^2$ not unif. contin. on all of $\mathbb{R}$,

is unif. contin. on any bounded interval.



$f(x)=1/x$ not unif contin. on all of $\mathbb{R}-\{0\}$.

Not unif contin on $(0,1)$,

is unif. contin on intervals away from 0.

We want:

"If $f(x)$ is continuous and K is ______ ?

then f is unif continuous on K."

open intervals are not allowed

unbounded intervals are not allowed.

Thm If f is (pointwise) continuous and K is a compact subset of the domain, then f is unif. continuous on K.

Pf Let f be continuous, For a contradiction assume f is not unif. continuous on K.

Then: $\exists\, x_n, y_n \in K$ s.t $|x_n - y_n| \to 0$

and $|f(x_n) - f(y_n)| \not\to 0$.

Actually, we can assume $|f(x_n) - f(y_n)|$ is bounded away from 0.

Since K is compact: x_n has a conv. subseq

x_{n_k}, and $x_{n_k} \to x \in K$.

Also $\exists\, y_{n_k}$, $y_{n_k} \to y \in K$.

Since $|x_n - y_n| \to 0$,

$|x_{n_k} - y_{n_k}| \to 0$

but also $|x_{n_k} - y_{n_k}| \to |x - y|$

So $0 = |x - y|$, i.e. $x = y$

So $|f(x_n) - f(y_n)|$ is bounded away from 0,

and $x_{n_k} \to x$ and $y_{n_k} \to x$.

Since f is continuous,

$$f(x_{n_k}) \to f(x) \text{ and } f(y_{n_k}) \to f(x)$$

So $|f(x_{n_k}) - f(y_{n_k})| \to |f(x) - f(x)| = 0$,

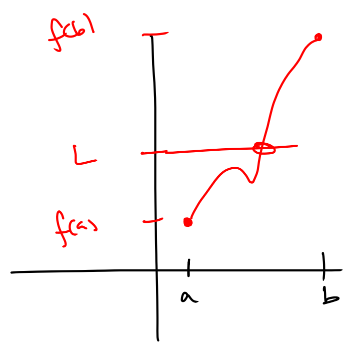
contradicts $|f(x_n) - f(y_n)|$ is bounded away from 0.

Intermediate Value Theorem

Thm let f be continuous on $[a,b]$.

If $f(a) < L < f(b)$ or $f(b) < L < f(a)$

then $\exists c \in [a,b]$ with $f(c) = L$.



Using IVT:

Thm $\sqrt{2}$ exists. ie. there is some $x \in \mathbb{R}$ with $x^2 = 2$.

Pf let $f(x) = x^2 - 2$

We want a real # $c \in \mathbb{R}$ which makes $f(c) = 0$.

$f(1) = -1$

$f(3) = 3^2 - 2 = 7$

Considering f on $[1, 3]$,

$f(1) < 0 < f(3)$

So by IVT $\exists c \in [1,3]$ s.t.

$f(c) = 0$.

So $c^2 - 2 = 0$

So $c^2 = 2$ Shown.

More generally: $\sqrt{K}$ exists for any $K \geq 0$.

let $f(x) = x^2 - K$

$f(0) = 0^2 - K = -K$ is negative

$f(K) = K^2 - K = K(K-1)$ sometimes pos, sometimes neg.

$f(k+1) = (k+1)^2 - k$

$= k^2 + 2k + 1 - k = k^2 + k + 1$ is positive since k is positive.

So on the interval $[0, k+1]$,

$$f(0) < 0 < f(k+1)$$

So $\exists c \in [0, k+1]$ with $f(c) = 0$

i.e. $c^2 - k = 0$

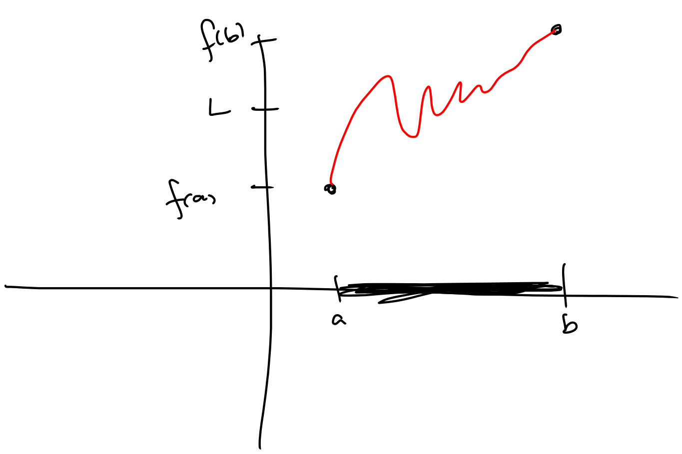
$c^2 = k$.

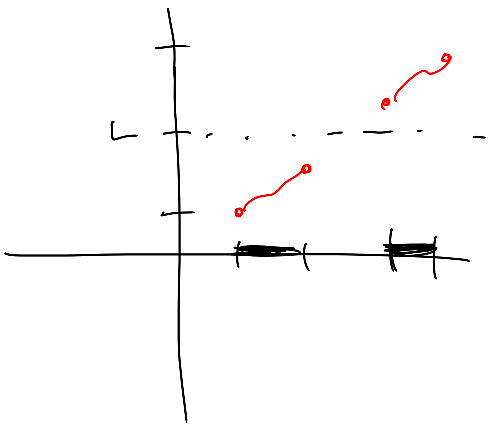
Special Kase $k=0$:

i.e. find c with $c^2 - k = 0$

i.e. $c^2 = 0$

$c = 0$ satisfies this.





Thm (IVT) If f is continuous and E is connected, then $f(E)$ is connected.