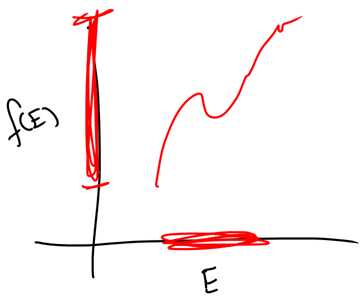
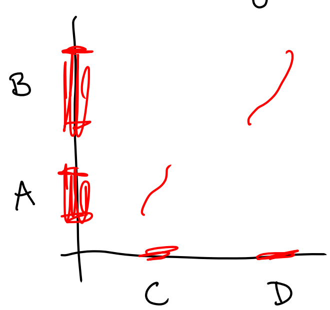
Thm If f is continuous and E is a connected set, then $f(E)$ is connected.



We'll prove the contrapositive:
Let f be continuous. If $f(E)$ is disconnected, then E is disconnected.

Pf Let f be continuous & $f(E)$ disconnected.

So $f(E) = A \cup B$ where $\bar{A} \cap B = \emptyset$
and $A \cap \bar{B} = \emptyset$



Let $C = \{x \in E \mid f(x) \in A\} = E \cap f^{-1}(A)$
$D = \{x \in E \mid f(x) \in B\} = E \cap f^{-1}(B)$

So $E = C \cup D$, must check $\bar{C} \cap D = \emptyset$ & $C \cap \bar{D} = \emptyset$.

FSOC, assume $x \in \bar{C} \cap D$.

since $x \in D$, $f(x) \in B$.

since $x \in \bar{C}$, either $x \in C$ or x is a lim pt of C.

$\uparrow$ $f(x) \in A$

$\uparrow$ $f(x)$ is a lim pt of A since f is continuous.

So $f(x) \in \bar{A}$

So $f(x) \in \bar{A} \cap B$ $\quad \rightarrow\leftarrow \quad$ $(\bar{A} \cap B = \emptyset)$