The Derivative

Def If $f: \mathbb{R} \to \mathbb{R}$ is a function and c is in the domain of f,

then <u>the derivative of f at c</u> is.

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

equiv. to $\displaystyle \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

If $f'(c)$ exists, we say f is <u>differentiable</u> <u>at</u> <u>c</u>.

Ex) $f(x) = x^n$ is differentiable at any $c \in \mathbb{R}$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c} \frac{x^n - c^n}{x - c}$$

$$x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + c^2 x^{n-3} + \cdots + c^{n-1})$$

So $$f'(c) = \lim_{x \to c} \frac{\cancel{(x-c)}(x^{n-1} + cx^{n-2} + \cdots + c^{n-1})}{\cancel{x - c}}$$

$$= \lim_{x \to c} x^{n-1} + cx^{n-2} + \cdots + c^{n-1}$$

$= c^{n-1} + c \cdot c^{n-2} + \cdots + c^{n-1}$

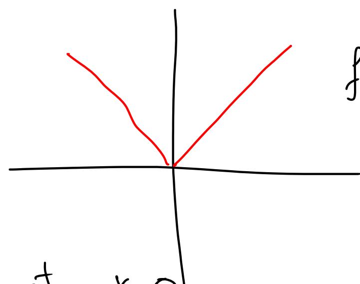
$= c^{n-1} + c^{n-1} + \cdots + c^{n-1} = nc^{n-1}$

So $\boxed{f'(c) = nc^{n-1}}$ so the limit exists

so f is differentiable at c.

$f'(x) = nx^{n-1}$

A nondifferentiable function:



$f(x) = |x|$

$f(x) = |x|$ is not differentiable at $x=0$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \underbrace{\frac{|x|}{x}}_{\begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}}$$

We'll show this limit does not exist with sequences:

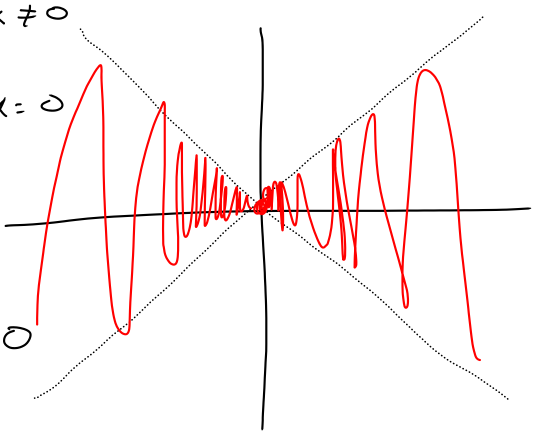
$x_n = 1/n \to 0$

$y_n = -1/n \to 0$

but $\frac{|x_n|}{x_n} \to 1$ and $\frac{|y_n|}{y_n} \to -1$

So $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



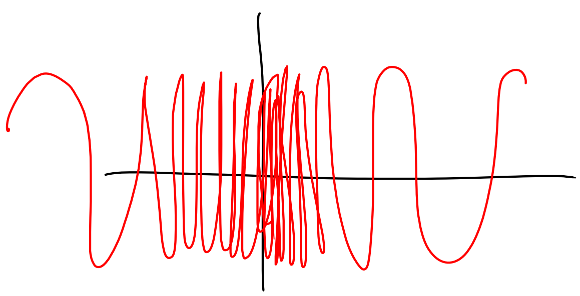
This is continuous at $x=0$

Is it differentiable at $x=0$?

$$f'(0) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0} \frac{x\sin(1/x)}{x} = \lim_{x\to 0} \sin(1/x)$$

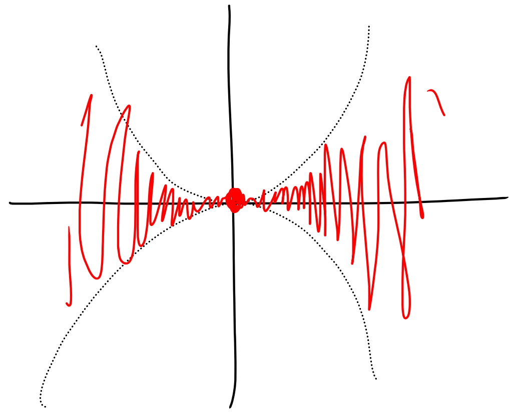
Does not exist.



So f is continuous at 0, but not differentiable at 0.

$$f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is f differentiable at 0?



$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x) - 0}{x - 0}$$

$$= \lim_{x \to 0} x \sin(1/x) = 0$$

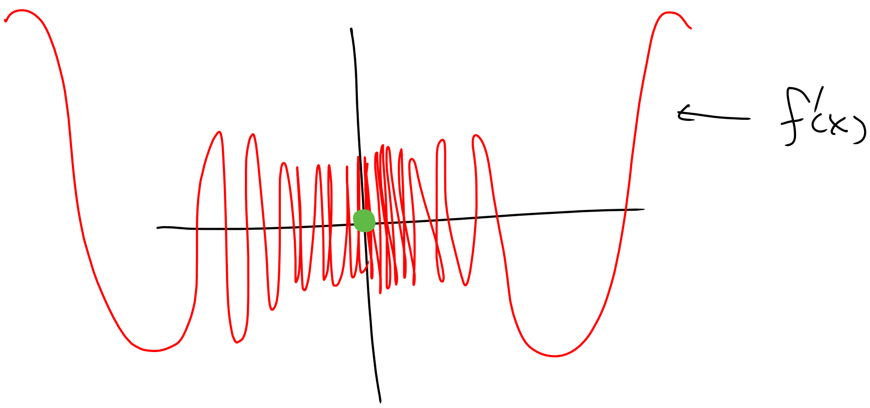
So $f'(0)$ exists   So f is differentiable at 0.

Using product & chain rule:

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \cancel{x^2} \cdot \cos(1/x) \cdot -\cancel{x^{-2}} + \sin(1/x) \cdot 2x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$= \begin{cases} -\cos(1/x) + 2x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



So f is differentiable at 0,

$f'(0)$ exists,

but f' is not continuous at 0.

(so $f''(0)$ does not exist)