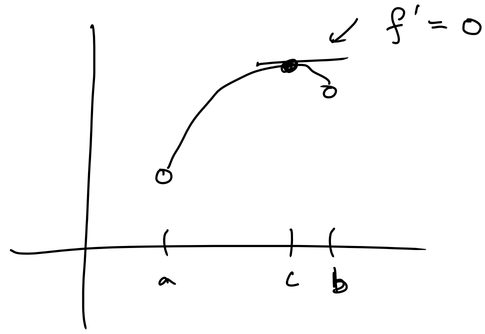
Relative extrema



Thm If f is differentiable on (a,b) and $f(c)$ is a maximum value of f on (a,b), then $f'(c) = 0$.

Pf $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ we'll show this is 0.

Use sequences: Take $x_n \to c$ with $x_n \le c$ $\forall n$
$y_n \to c$ with $y_n \ge c$ $\forall n$.

so $x_n \le c \le y_n$

Since f is continuous,
$f(x_n) \to f(c)$ and $f(x_n) \le f(c)$
and $f(y_n) \to f(c)$ and $f(y_n) \le f(c)$
since $f(c)$ is the max.

So $f'(c) = \lim \dfrac{f(x_n) - f(c)}{x_n - c}$ $\leftarrow$ neg $\leftarrow$ neg

so $f'(c) \ge 0$

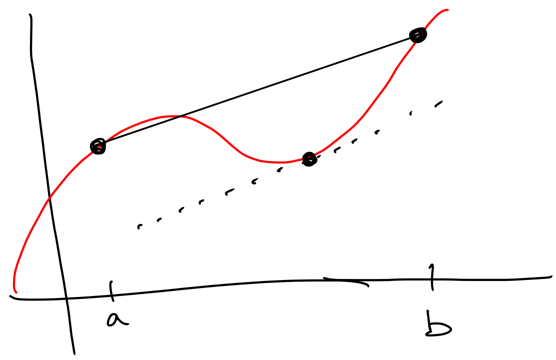
also: $f'(c) = \lim \dfrac{f(y_n) - f(c)}{y_n - c}$ $\begin{array}{l}\leftarrow \text{neg} \\ \leftarrow \text{pos}\end{array}$ $\boxed{\text{so } f'(c) \le 0}$

So $f'(c) = 0$.

Sham.

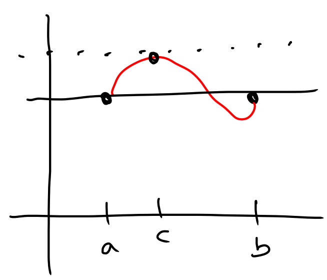
Mean Value Theorem

Some interior point has slope equaling the average (mean) slope.



Baby MVT: "Rolle's Theorem"

Thm If f is differentiable on $[a,b]$, and $f(a) = f(b)$, then $\exists c \in (a,b)$ such that $f'(c) = 0$.



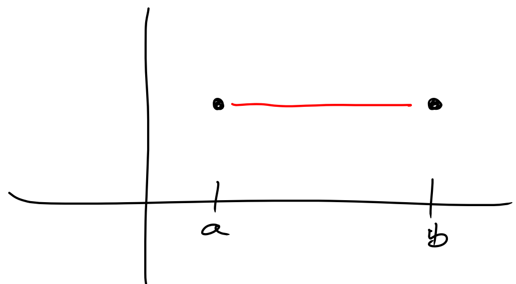
Pf $[a,b]$ is compact, so there $\exists$

a min & max value of f on $[a,b]$.

If some interior point c is a min or max, then $c \in (a,b)$ and we use the other thm to say $f'(c) = 0$ as desired.

In the special case when $f(a)$ & $f(b)$ are the min & max,

Then f must be a constant function,



So $f'(c) = 0$ for all $c \in (a,b)$.

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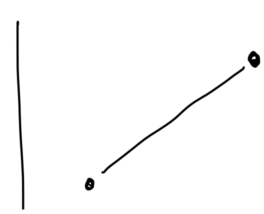
2 cute corollaries:

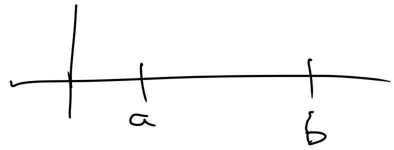
<u>Cor</u> If $f'(x) = 0$ for all x, then $f(x) = c$ for all x.

<u>Pf</u> Let $f'(x) = 0$ $\forall x$, assume FSOC that $f(a) \neq f(b)$ for some $a, b \in \mathbb{R}$.

So $\exists c$ between a & b with $f'(c) \neq 0$



$\Rightarrow f'(x) = 0 \ \forall x.$



<u>Cor</u> If f & g are differentiable and $f' = g'$, then: $f(x) = g(x) + K$. for some $K \in \mathbb{R}$.

<u>Pf</u> Let $h(x) = f(x) - g(x)$.

Then $h'(x) = f'(x) - g'(x) = 0$

So $h(x) = K$

So $K = f(x) - g(x)$, so

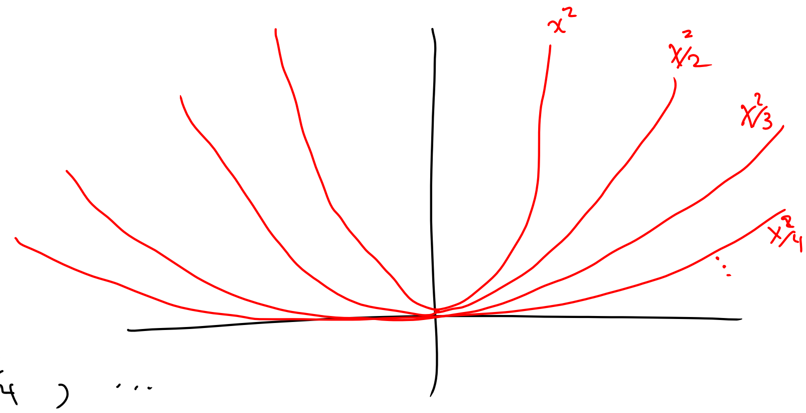
$f(x) = g(x) + K$ Shown.

Sequences of functions

If (f_n) is a sequence of functions, can we discuss $f_n \to f$? yes

Fun examples:

$$f_n(x) = \frac{x^2}{n}$$

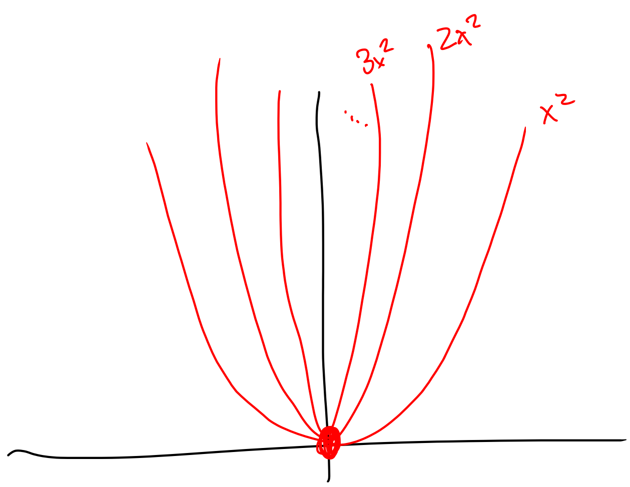


$x^2, \frac{x^2}{2}, x^2/3, x^2/4, \ldots$

Looks like $f_n \to f(x)=0$

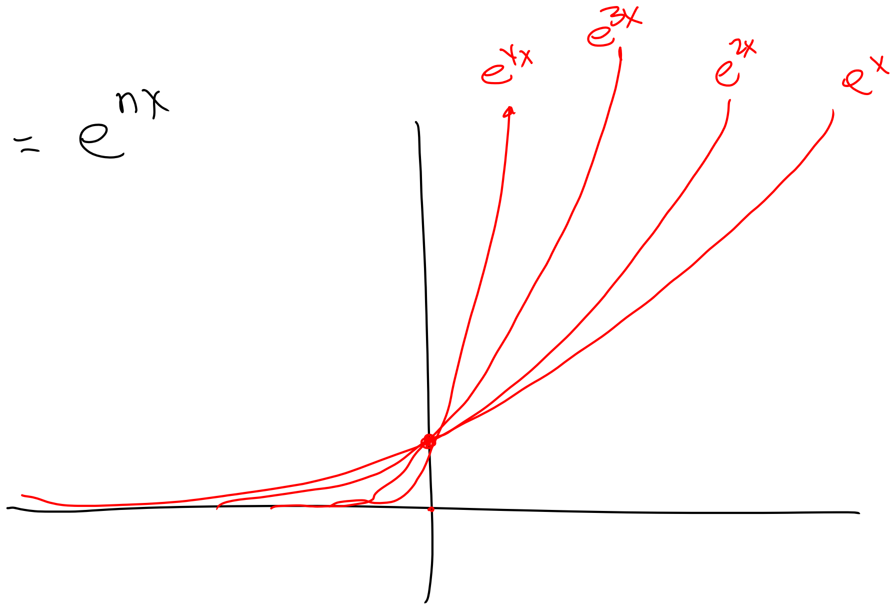
Note: f_n are unbounded for each n, but converge to 0 (bounded)

$$f_n(x) = nx^2$$



f_n does not converge to a function.
although $f_n(0) = 0$

$f_n(x) = e^{nx}$



this seems to converge to

$$\begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \\ \text{nothing} & \text{if} \quad x > 0 \end{cases}$$