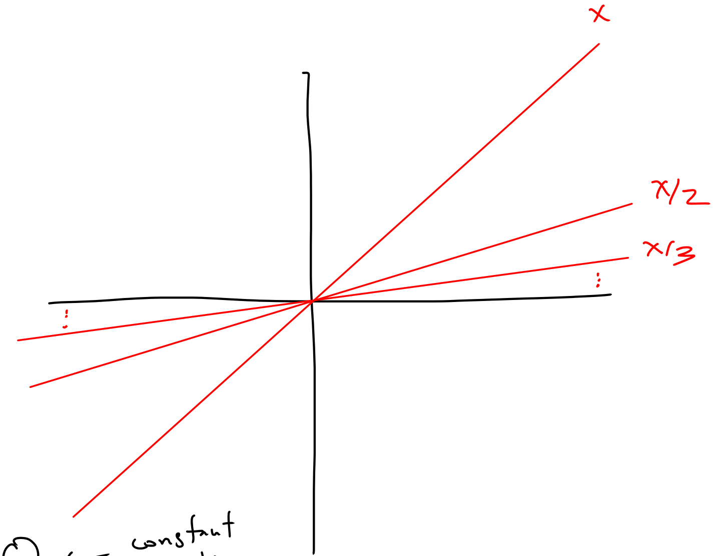
Sequences of functions

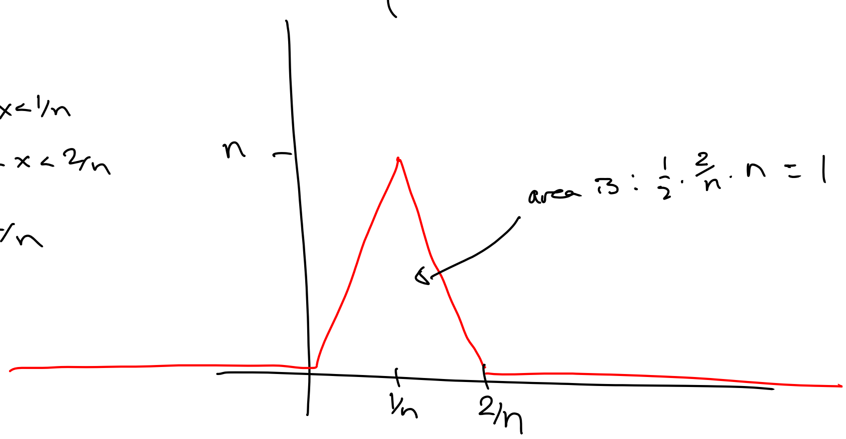
$$f_n(x) = \frac{x}{n}$$

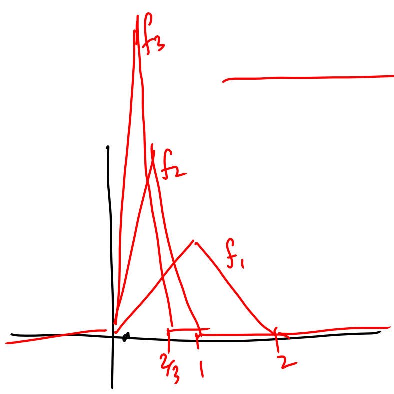
$x, \frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \ldots$



here, $f_n(x) \to 0$ ← constant function.

$$f_n(x) = \begin{cases} 0 & \text{if } x \le 0 \\ n^2 x & \text{if } 0 < x < 1/n \\ ??? & \text{if } 1/n < x < 2/n \\ 0 & \text{if } x \ge 2/n \end{cases}$$





For any $x > 0$, when n is big enough, $f_n(x) = 0$.

$f_n(x) \to 0$ for any x.

Here, $\int_{-\infty}^{\infty} f_n(x)\,dx = 1$

but $\int_{-\infty}^{\infty} 0\,dx = 0$.

Convergence of Seqs of functions

Pointwise conv. & Uniform convergence

weird stuff can happen

no weird stuff happens

[Pointwise] convergence

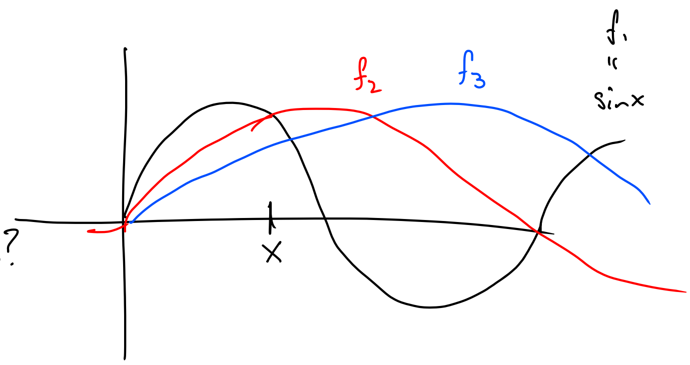
Def If f_n is a sequence of functions and A is a subset of the domain,

" $f_n \to f$ on A " means:

for each $x \in A$, $f_n(x) \to f(x)$

Ex $f_n(x) = \sin\left(\frac{x}{n}\right)$

What does f_n converge to?



For some some x?

$\frac{x}{n} \to 0$

$x \to a$

then $f(x) \to f(a)$

$f_n(x) \to$?

$\sin\left(\frac{x}{n}\right) \to \sin(0)$ since sin is continuous.

$= 0$

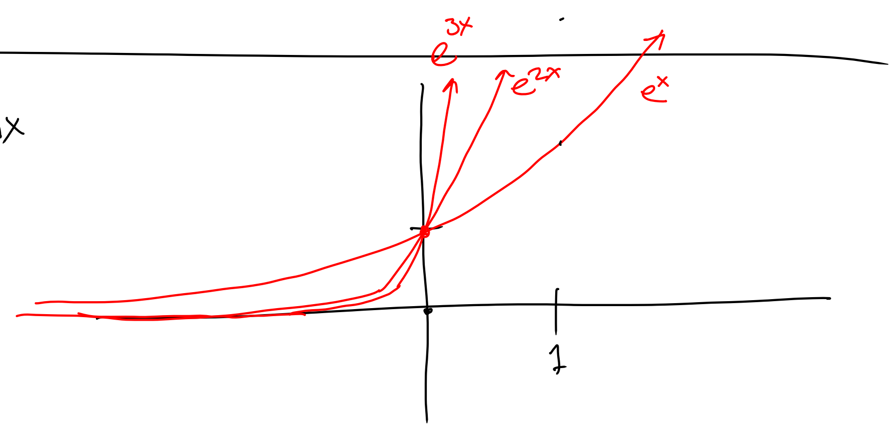
So it seems $f_n \to 0$.

Pf let $x \in \mathbb{R}$, then

$f_n(x) = \sin\left(\frac{x}{n}\right) \to 0$ since sin is continuous.

so $f_n(x) \to f(x)$ where $f(x) = 0$.

$f_n(x) = e^{nx}$



Let's show $f_n \to f$ where

$$f = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$f : (-\infty, 0] \to \mathbb{R}$

Pf First take $x = 0$: WTS $f_n(0) \to f(0)$

$$f_n(0) = e^{n\cdot 0} = e^0 = 1$$

$$f(0) = 1 \quad \text{so} \quad \underset{1}{\underset{\|}{f_n(0)}} \to \underset{1}{\underset{\|}{f(0)}}$$

Now consider $x<0$: WTS $f_n(x) \to f(x)$

$$f_n(x) = e^{nx}$$

$$f(x) = 0$$

when $x<0$, is $e^{nx} \to 0$? Yes!

e^{nx} is like $\dfrac{1}{e^{n\cdot ?}} \to 0$ as $n \to \infty$.

So $f_n \to f$ pointwise on $(-\infty, 0]$

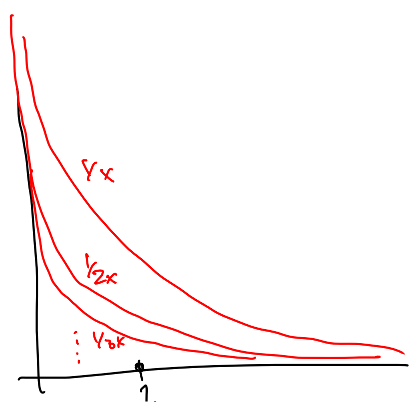
Uniform Convergence

Def f_n converges uniformly to f on A when:

$\forall \varepsilon > 0$ $\underbrace{\exists N \in \mathbb{N}}$ s.t. $\underbrace{\forall x \in A}$, ← Same N works for every x in the domain.

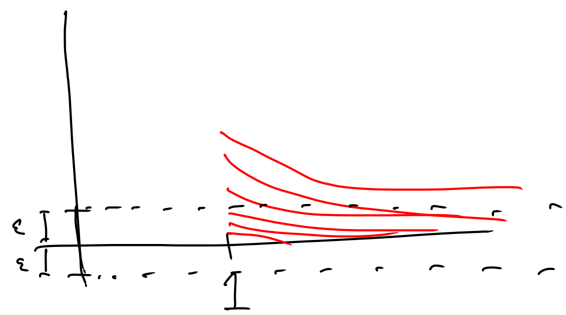
$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

Ex $f_n(x) = \dfrac{1}{n\cdot x}$



This does not converge uniformly on $(0,\infty)$

It does converge unif. on $[1,\infty)$



Unif. conv. means eventually, the entire function lies inside the ε-strip.

Let $\varepsilon>0$ be given, we'll find $N\in\mathbb{N}$ s.t. $\forall x\in[1,\infty)$,

$$n>N \Rightarrow |f_n(x)-f(x)|<\varepsilon$$

$$|f_n(x)-f(x)| = \left|\frac{1}{nx}-0\right| = \frac{1}{nx} < \frac{1}{n}$$

want $\frac{1}{n}<\varepsilon$