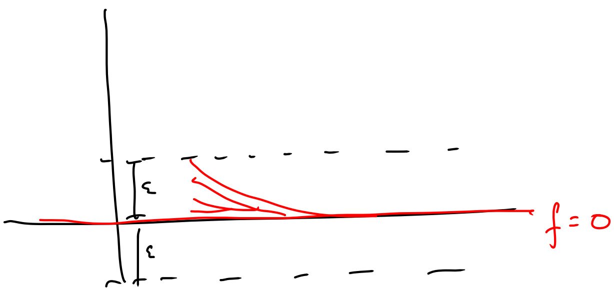
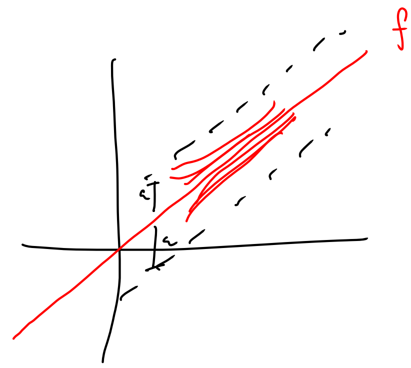
Uniform Convergence of seqs of functions

Def $f_n \to f$ uniformly on A when:

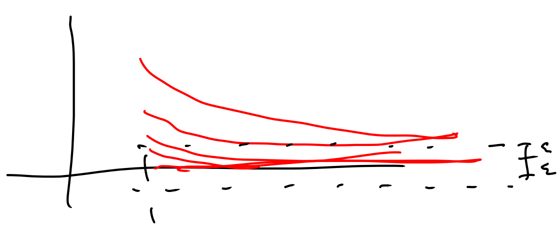
$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall x \in A$,

$$n > N \implies |f_n(x) - f(x)| < \varepsilon.$$

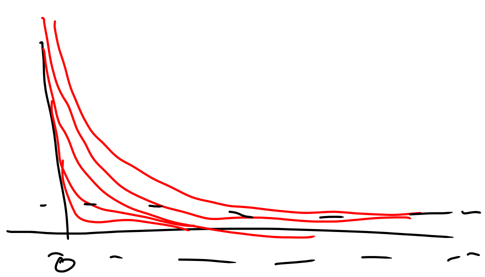




on $(1, \infty)$ $\frac{1}{nx} \to 0$ uniformly



On $(0, \infty)$, it's not uniform



$\underline{\text{Ex}}$ $\quad f_n(x) = \frac{1}{nx}$, WTS $f_n \to 0$ unif. on $(1,\infty)$

$\underline{\text{Pf}}$ let $\varepsilon > 0$ be given we'll find $N \in \mathbb{N}$ s.t. $\forall x \in (1,\infty)$

$$n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

$$\left[|f_n(x) - f(x)| = \left|\frac{1}{nx}\right| = \frac{1}{nx} < \frac{1}{n} \right.$$

since $x>1$

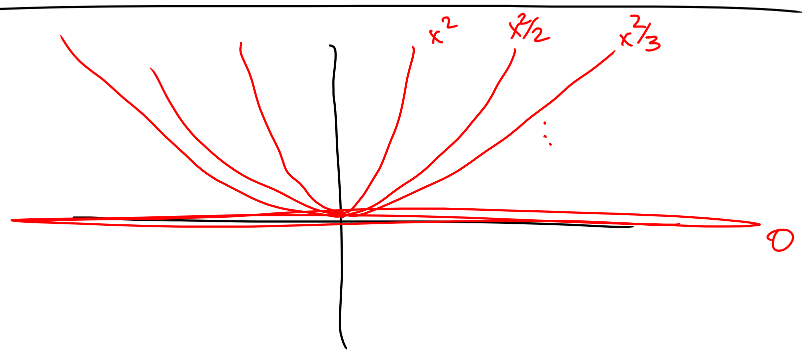
we want $\frac{1}{n} < \varepsilon$, so $n > \frac{1}{\varepsilon}$

Let $N > 1/\varepsilon$, then if $n > N$, we have:

$$|f_n(x) - f(x)| = \frac{1}{nx} < \frac{1}{n} < \frac{1}{1/\varepsilon} = \varepsilon$$

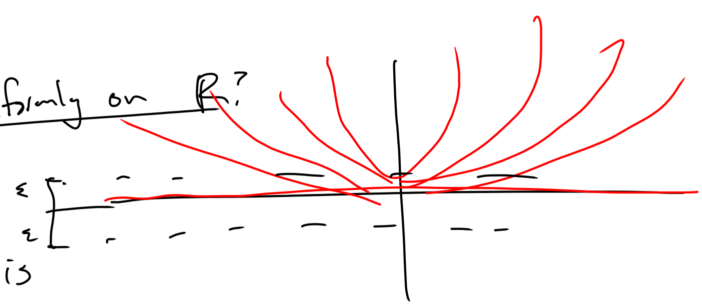
Sham.

$f_n(x) = \frac{x^2}{n}$



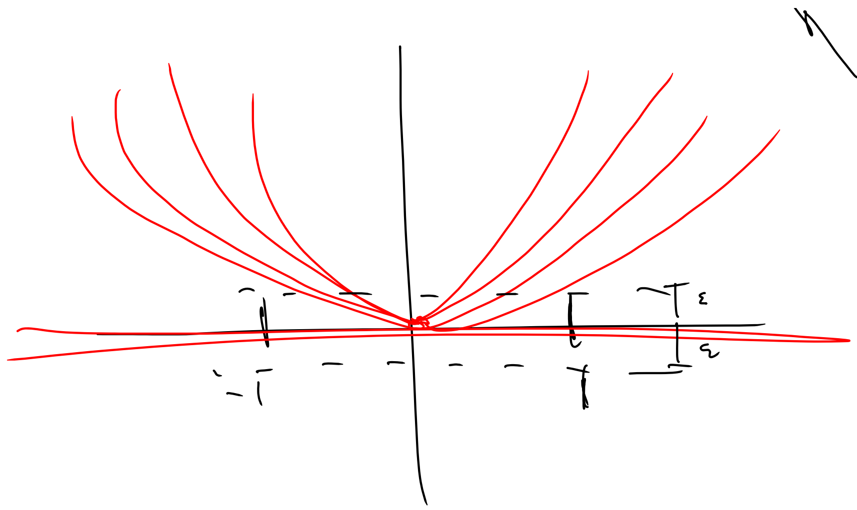
Pointwise, $f_n(x) \to 0$. $\forall x$.

does $f_n(x) \to 0$ uniformly on $\mathbb{R}$?



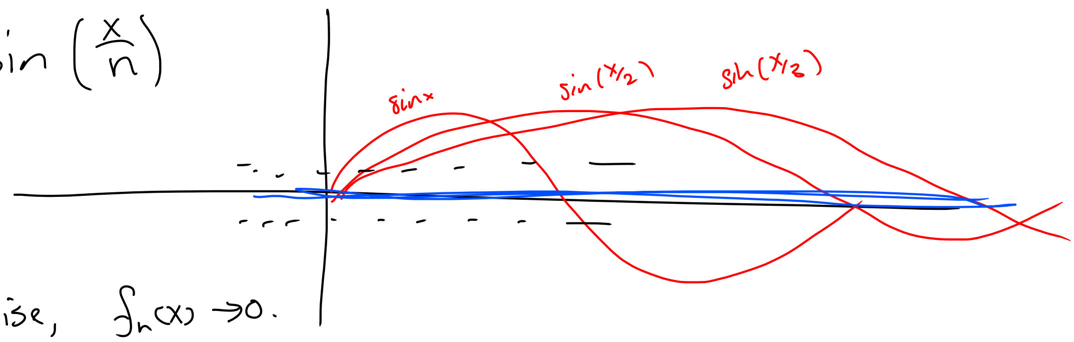
NO - each f_n is a parabola, so none is within ε of 0 on all of $\mathbb{R}$.

does $f_n(x) \to 0$ unif. on $[-1,1]$?



Yes!

$f_n(x) = \sin\left(\frac{x}{n}\right)$



Pointwise, $f_n(x) \to 0$.

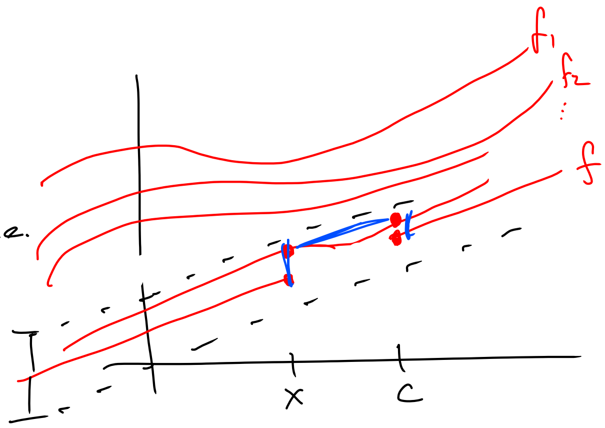
On $\mathbb{R}$, is it converging uniformly? NO

but on a bounded interval, it does converge uniformly.

Nice theorems are true when the convergence is uniform.

- If f_n is continuous, then f is continuous
- If f_n is bounded, ... f is bounded.
- If f_n is differentiable ... f is differentiable
- If f_n is integrable ... f is integrable.

Thm If $f_n \to f$ uniformly on A and f_n are all continuous on A, then f is continuous on A.



Pf Take $c \in A$, we'll show f is continuous at c, i.e.

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

Let $\varepsilon > 0$ be given, we'll find $\delta > 0$ s.t.

$$0 < |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

we'll do $\varepsilon/3$!

$$|f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)|$$

So $|f(x) - f(c)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$

choosing n large enough & δ small enough, we can make:

$|f(x) - f_n(x)| < \varepsilon/3$ since $f_n \to f$ on A. (uniformly!)

$|f_n(c) - f(c)| < \varepsilon/3$ " "

$|f_n(x) - f_n(c)| < \varepsilon/3$ since f_n is continuous

So $|f(x) - f(c)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ Shown!