Fun with series!

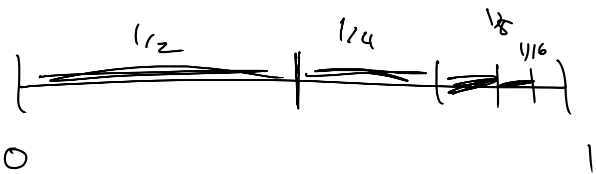
A **series** is a sum $\sum a_n$

where a_n is a sequence.

Sometimes it converges, sometimes not.

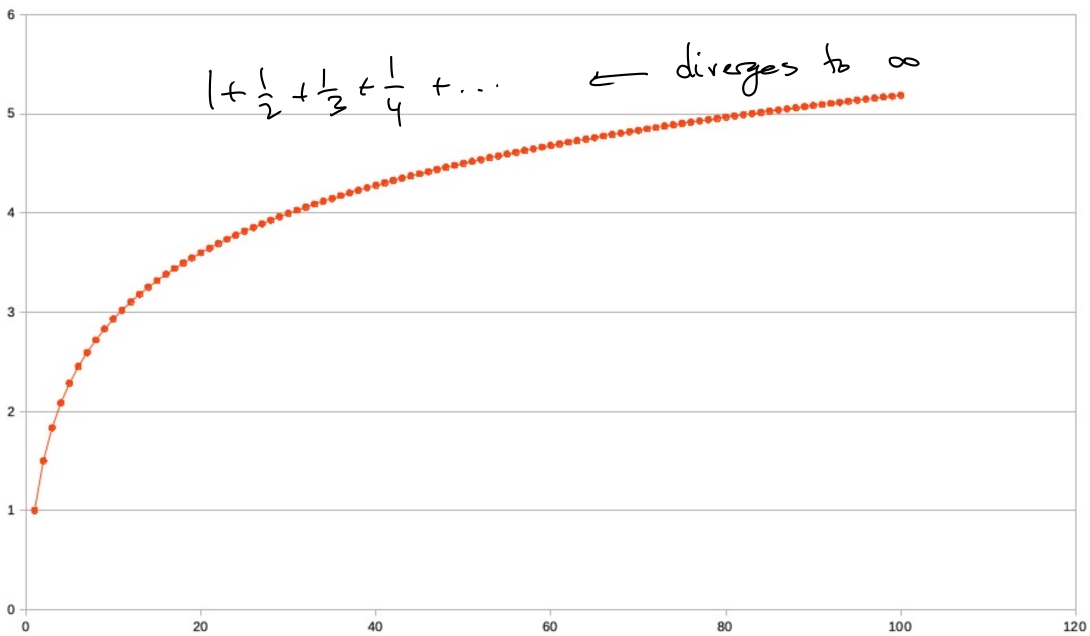
$$\sum \frac{1}{2^n} \leftarrow \text{"geometric series"}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$



$$\sum \frac{1}{n} \leftarrow \text{the harmonic series}$$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \leftarrow$ diverges to ∞



$\sum \frac{1}{n}$ is defined as:

let $s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ ← the nth partial sum

$\sum \frac{1}{n} = \lim s_n$ ← equals ∞:

Why s_n is unbounded:

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

if $n = 2^k$

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots \left(+ \frac{1}{2^k}\right)$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \dots + \left(\dots + \frac{1}{2^k}\right)$$

$$= 1 + \frac{1}{2} + \left(1/2\right) + \left(\frac{1}{2}\right) + \dots + \left(\frac{1}{2}\right)$$

$$= 1 + \frac{1}{2} \cdot k$$

so $S_{2^k} \geq 1 + k/2$, so S_{2^k} are unbounded

so S_n is unbounded.

so $\sum \frac{1}{n} = \infty$

The Alternating Harmonic Series

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots$$

$$\sum \frac{(-1)^n}{n}$$ "alternating harmonic"

$\sum \frac{1}{n}$ diverges. $\quad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

just the evens: $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots = \sum \frac{1}{2n}$

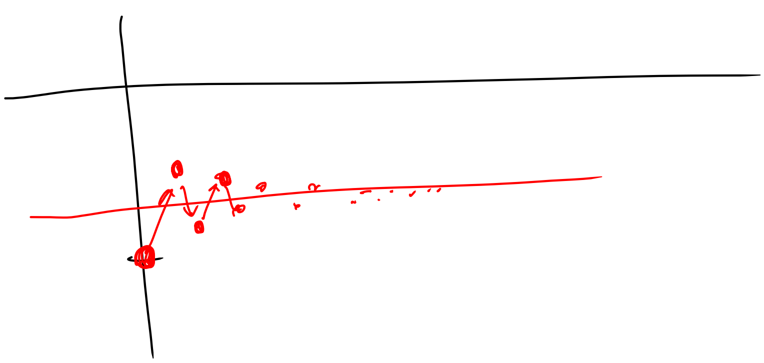
$\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ so $\sum \frac{1}{2n}$ also diverges.

just the odds also diverges.

In $-1+\boxed{\frac{1}{2}-\frac{1}{3}}+\boxed{\frac{1}{4}-\frac{1}{5}}+\ldots$

just the evens make it $\to \infty$

but just the odds make it $\to -\infty$



Turns out $\sum \frac{(-1)^n}{n} = -.69\ldots$

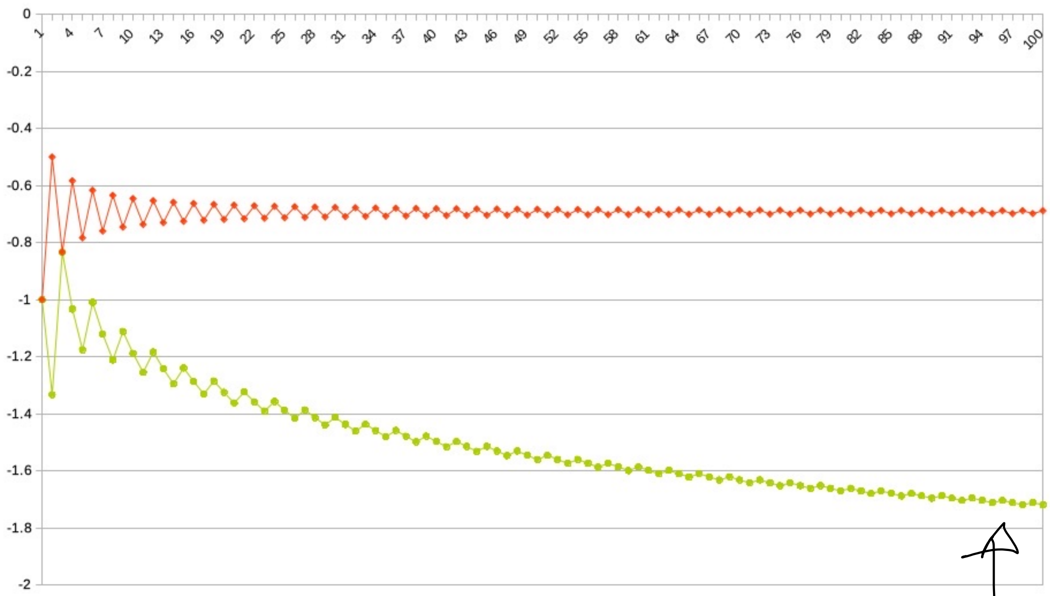
$-\ln 2$

They balance out! But it's delicate:

What if we add 2 odds at a time?

$$\boxed{-1 - 1/3 + 1/2 - 1/5 - \frac{1}{7} + \frac{1}{4} - \frac{1}{9} - \frac{1}{11}} + \frac{1}{6} + \ldots$$

Each partial sum S_n will be less than before!

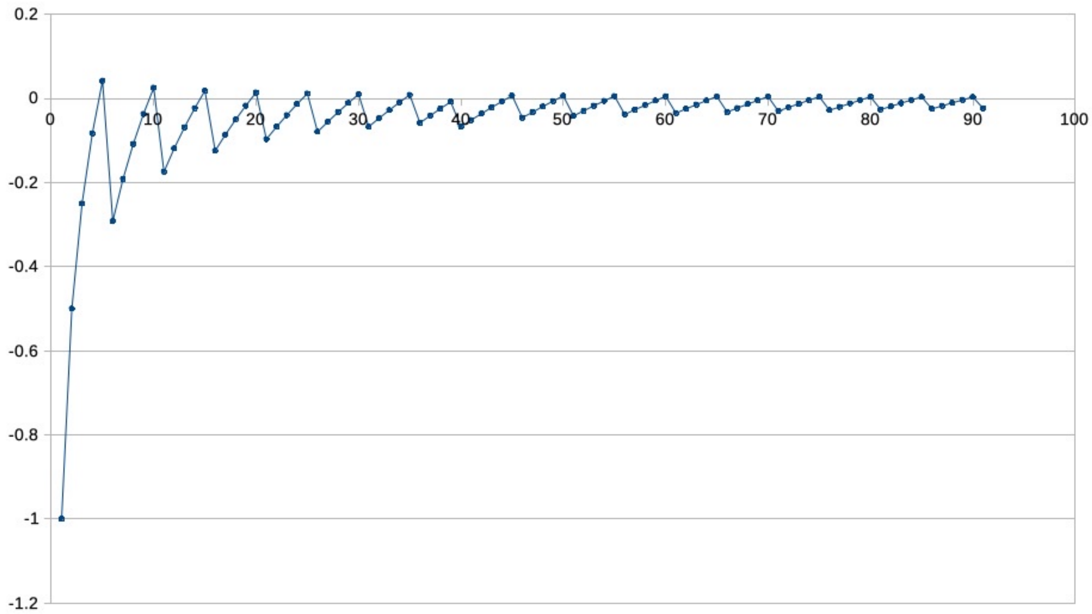


In an infinite sum, rearranging the terms can change the value

$$1-7 \neq 7-1$$

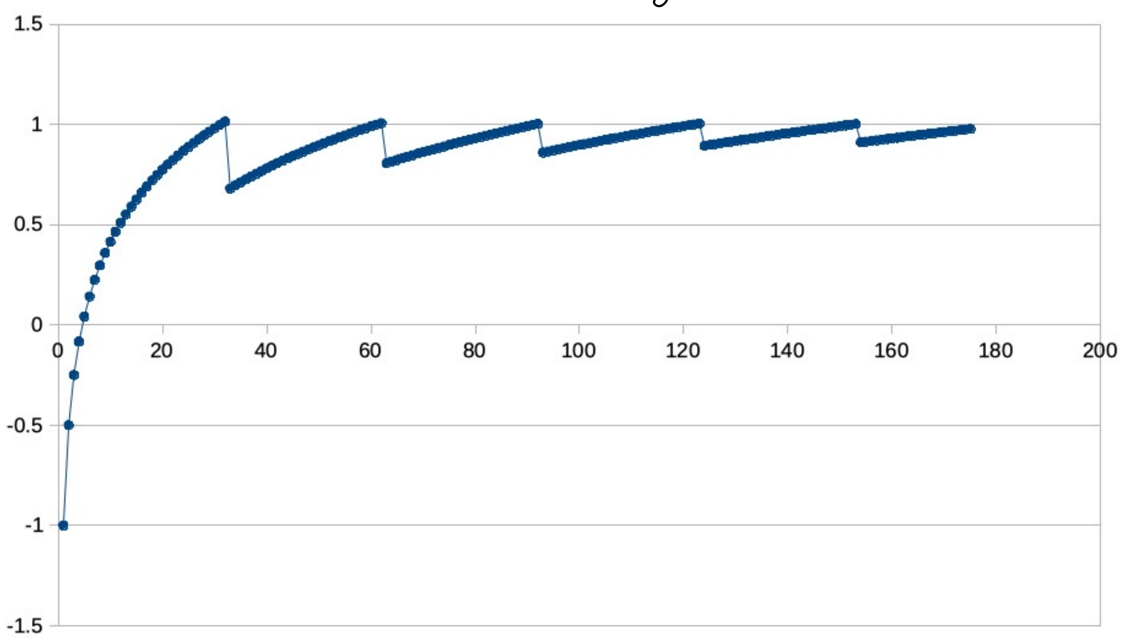
$$1+(-7) = -7+1$$

2 odds for every even will diverge to $-\infty$.



This one converges to 0.

converges to 1!



Thm For any $c \in \mathbb{R}$, or $c = \infty$ or $c = -\infty$,

there is some rearrangement of $\sum (-1)^n/n$

which converges to c.

(This is true for any series where $\sum |a_n|$ diverges but $\sum a_n$ converges)