Math 3371 first exam topics & practice

Bounds, sup, inf, completeness

1. According to definitions, is $\emptyset$ bounded above? If so, give an upper bound.
2. Give examples of a set S with $\sup S \in S$. Give an example where $\sup S \notin S$. Same with inf. Do all these without using intervals.
3. Give an example where $\sup S$ does not exist. Same with inf.
4. Give an example of a bounded subset of $S \subseteq \mathbb{Q}$ with $\sup S \notin \mathbb{Q}$.
5. Let $S = \{\sin n \mid n \in \mathbb{N}\}$. Show that $\sup S$ and $\inf S$ exist, and say what they are. (don't prove what they are)

Nested intervals, Archimedean property

6. Give examples of infinitely many nested closed intervals whose intersection is: a closed interval, a single point, all of $\mathbb{R}$, the empty set. For each one, either give an example or show that it's impossible.
7. Explain how the Archimedian property implies that there is no "biggest real number".

Cardinality

8. Show that the set of multiples of 7 has the same cardinality as $\mathbb{N}$.
9. Give an example of an uncountable set other than $\mathbb{R}$ or an interval.

Convergence

10. Show that these converge: $\frac{2n^3+1}{n^3}$, $\frac{2n}{n^3+1}$, $\frac{2n+1}{3n^3}$. Do these with limit rules (easy), or using the definition of convergence (harder).
11. Assume that $(x_n) \to 3$, and show $(2x_n + 1) \to 7$. (Again, try using limit rules, and also from the definition.)

Monotone convergence theorem, Bolzano-Weierstrauss theorem

12. Invent an example sequence for which the monotone convergence theorem applies.
13. Invent an example sequence which is bounded but not convergent. What does the Bolzano-Weierstrauss theorem say about that sequence?
14. Use the Bolzano-Weierstrauss theorem (contrapositive) to show this sequence diverges: (1, 1/2, 1, 1/3, 1,1/4,1,1/5,...)

Open & closed sets

15. Give examples of an open set, a closed set, and a set which is neither open nor closed. Don't use intervals.

16. Prove from the definition that $(0, 1)$ is open.

17. Prove from the definition that 3 is a limit point of $\mathbb{Q}$.

18. If A and B are both not open, is $A \cup B$ not open? How about $A \cap B$? Prove your answer either way.

Answers!

1. It is bounded above by any real number.
2. $S = \{1, 2\}$ has $\sup S \in S$.
   $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ has $\sup S \notin S$.
3. $S = \mathbb{R}$ has no sup or inf.
4. $S = \mathbb{Q} \cap (0, \sqrt{2})$.
5. S is bounded above and below, so the sup and inf exist by the Axiom of Completeness. They are 1 and -1.
6. Closed interval: $\{[-\frac{1}{n}, 1 - \frac{1}{n}]\}$. Single point: $\{[-\frac{1}{n}, \frac{1}{n}]\}$. All of $\mathbb{R}$ is possible only when every interval is all of $\mathbb{R}$, which doesn't really count as closed intervals. The empty set is impossible because NIP says the intersection must be nonempty.
7. The Archimedean property (big version) says: "For any $x \in \mathbb{R}$, there is some $n \in \mathbb{N}$ with $n > x$." This means that there can be no biggest real number: if you think that x is the biggest real number, then immediately we find some $n > x$ which is a contradiction.
8. $f(x) = 7x$ is a bijection from $\mathbb{N}$ to the set of multiples of 7, so these sets have the same cardinality.
9. The set of irrational numbers is uncountable. Or something like $[1, 2] \cup [2, 3]$ which is uncountable but not an interval.
10. $\frac{2n^3+1}{n^3} \to 2$

   Proof. Let $\varepsilon > 0$ be given, we will find $N \in \mathbb{N}$ such that, when $n > N$, we have

   $$\left|\frac{2n^3+1}{n^3} - 2\right| < \varepsilon.$$

   We simplify:

   $$\left|\frac{2n^3+1}{n^3} - 2\right| = \left|\frac{1}{n^3}\right| = \frac{1}{n^3}$$

   I want $\frac{1}{n^3} < \varepsilon$, so I need $n > \frac{1}{\sqrt[3]{\varepsilon}}$.

   Let $N > \frac{1}{\sqrt[3]{\varepsilon}}$. Then if $n > N$ we have:

   $$\left|\frac{2n^3+1}{n^3} - 2\right| = \frac{1}{n^3} < \frac{1}{N^3} < \frac{1}{1/\sqrt[3]{\varepsilon^3}} = \varepsilon$$

   as desired. □

   $\frac{2n}{n^3+1} \to 0$

   Proof. For this one I'll just do the red part– I will expect you to write out the whole proof on the test.

   $$\left|\frac{2n}{n^3+1} - 0\right| = \frac{2n}{n^3+1} < \frac{2n}{n^3} = \frac{2}{n^2}$$

   so we need $n > \sqrt{2/\varepsilon}$. □

   $\frac{2n+1}{3n^3} \to 0$

Proof. For this one I'll just do the red part– I will expect you to write out the whole proof on the test.

$$\left|\frac{2n+1}{3n^3} - 0\right| = \frac{2n+1}{3n^3} < \frac{2n+n}{3n^3} = \frac{3n}{3n^3} = \frac{1}{n^2}$$

so we need $n > \frac{1}{\sqrt{\varepsilon}}$. □

11. $(2x_n + 1) \to 7$

Proof. Let $\varepsilon > 0$ be given, we will find $N \in \mathbb{N}$ such that, when $n > N$, we have

$$|2x_n + 1 - 7| < \varepsilon.$$

$$|2x_n + 1 - 7| = |2x_n - 6| = 2|x_n - 3|$$

I want this to be less than ε, so I need $|x_n - 3| < \varepsilon/2$.

Since $x_n \to 3$, we may choose N so big that $n > N$ implies $|x_n - 3| < \varepsilon/2$. Then if $n > N$ we have:

$$|2x_n + 1 - 7| = 2|x_n - 3| < 2\frac{\varepsilon}{2} = \varepsilon$$

as desired. □

12. $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is monotone (decreasing) and bounded, so it converges.

13. $\{(-1)^n\}$ is bounded but not convergent. B-W theorem says that there is a convergent subsequence.

14. I can find 2 subsequences converging to two different limits: one is $(1, 1, 1, 1, \dots)$ which converges to 1, and the other is $(\frac{1}{n})$, which converges to 0.

15. $(0, 1) \cup (2, 3)$ is open. $\{1, 2, 3\}$ is closed. $\{\frac{1}{n}\}$ is neither open nor closed.

16. Choose any $x \in (0, 1)$, and we need to find some $\varepsilon > 0$ with $V_\varepsilon(x) \subseteq (0, 1)$. Draw a picture, it is clear that letting $\varepsilon = \min(x, 1 - x)$ will do it.

17. We must show that, for any $\varepsilon > 0$, the neighborhood $V_\epsilon(3) \cap \mathbb{Q}$ contains some points other than 3. Since $\mathbb{Q}$ is dense, this means that between any two reals is a rational. Thus there is a rational number q with $3 < q < 3 + \varepsilon$, and so $q \in V_\epsilon(3)$ as desired.

18. If A and B are both not open, then $A \cup B$ might be open: for example let $A = (0, 1]$ and $B = [1, 2)$. Then $A \cup B = (0, 2)$ which is open.

    If A and B are both not open, then $A \cap B$ might be open: for example let $A = [0, 2)$ and $B = (1, 3]$. Then $A \cap B = (1, 2)$ which is open.