# Math 3371 second exam topics & practice

## Closure of a set

1) Find the closure of these sets (don't have to prove it): $(0,1), [0,1], \{1,2,3\}, (0,1)\cap\mathbb{Q}, \{1/n \mid n \in \mathbb{N}\}, \mathbb{Q}-\mathbb{N}$

## Compact sets

2) Show that a closed interval is compact. (Be able to do this using sequences, or with the Heine-Borel theorem.)
3) Show that an open interval is not compact.
4) Show that $(0,1)$ is not compact by giving an open cover with no finite subcover.

## Connected sets

5) Show that $\mathbb{R}-\mathbb{Q}$ is not connected.
6) Show that:

$$\bigcup_{n\in\mathbb{N}}(2n, 2n+1)$$

is not connected.

7) Prove from the definition that a set of a single point is connected.
8) Is $\emptyset$ connected? Prove your answer.

## Limits of functions

9) Show that $\lim_{x\to 3}\frac{x+1}{x-2} = 4$. (Be able to do this using the $\epsilon$-$\delta$ definition, or using algebraic limit rules.)
10) Show that $\lim_{x\to -2}\frac{x+3}{x^2+x-2}$ does not exist.
11) Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, (and prove it) where:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

## Continuity

12) Find a point where $\frac{x+1}{x-2}$ is continuous, and prove it.

13) Find a point where this function is discontinuous, and prove it:

$$f(x) = \begin{cases} 3x+5 & \text{if } x \leq 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

14) Show that $[[x]]$ is discontinuous at every $c \in \mathbb{Z}$, and continuous at every $c \notin \mathbb{Z}$.

## Uniform Continuity

15) Give an example of a function $f$ such that $f$ is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$ (and prove it).

16) Show that $\frac{x+1}{x-2}$ is not uniformly continuous on $\mathbb{R} - \{2\}$.

17) Show that $x^2 - 6x + 3$ is uniformly continuous on $[-5, -1]$.

# Answers!

1) $[0,1], [0,1], \{1,2,3\}, [0,1], \{1/n \mid n \in \mathbb{N}\} \cup \{0\}, \mathbb{R}$

2) Using Heine-Borel: a closed interval is closed and bounded, so it's compact.

   Using sequences is a lot harder. Let $x_n \in [a,b]$, we'll show there is a subsequence which converges to some point in $[a,b]$. Since $x_n$ is bounded, by Bolzano-Weierstrauss there is a convergent subsequence, say $x_{n_k} \to x$. We need to show $x \in [a,b]$. Since all $x_n \in [a,b]$, we have $x_n \leq b$ for all $n$, so $x_{n_k} \leq b$ for all $k$, and since $x_{n_k} \to x$ we will have $x \leq b$ by the order-limit rules. For the same reason we will have $x \geq a$ so $x \in [a,b]$ as desired.

3) Using Heine-Borel: an open interval is not closed, so it's not closed-and-bounded, so it's not compact.

   Using sequences: For the open interval $(a,b)$, let $x_n = a + \frac{1}{n}$. Then $x_n \in (a,b)$ (when $n$ is large enough) but $x_n \to a$, so all subsequences converge to $a$, which is not an element of $(a,b)$. So $x_n$ has no subsequence converging to an element of $(a,b)$, so $(a,b)$ is not compact.

4) $O_x = (0,x)$ for $x \in (0,1)$ gives an open cover with no finite subcover.

5) Let $A = (-\infty,0) - \mathbb{Q}$ and $B = (0,\infty) - \mathbb{Q}$. Then $A \cup B = \mathbb{R} - \mathbb{Q}$, and $\bar{A} = (-\infty, 0]$ and $\bar{B} = [0,\infty)$ so $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

6) Let $A = (2,3)$, and $B = \bigcup_{n\geq 2}(2n, 2n+1)$. Then check all the details.

7) Say our set is $\{x\}$. We prove by contradiction: assume that $\{x\}$ is disconnected, so we have $\{x\} = A \cup B$ where $A$ and $B$ are nonempty and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. But if $A$ and $B$ are nonempty with $A \cup B = \{x\}$, then we must have $A = B = \{x\}$. And this would mean $\bar{A} \cap B \neq \emptyset$ which is a contradiction.

8) The empty set is connected. We will prove this by contradiction. Assume that $\emptyset$ is disconnected, so $\emptyset = A \cup B$ where $A$ and $B$ are nonempty and .... This immediately is a contradiction, since $\emptyset$ cannot be a unioin of nonempty sets.

9) Using the limit rules is easy.

   Using the definition: Let $\varepsilon > 0$ be given, we will find $\delta > 0$ such that:

$$0 < |x-3| < \delta \implies \left|\frac{x+1}{x-2} - 4\right| < \varepsilon.$$

   We simplify:

$$\left|\frac{x+1}{x-2} - 4\right| = \left|\frac{-3x+9}{x-2}\right| = 3|x-3| \cdot \frac{1}{|x-2|}$$

   Now if $\delta < 1/2$ we have $2.5 < x < 4.5$, so $x - 2 > 1/2$ so $\frac{1}{|x-2|} < 2$. Thus the above becomes

$$\left|\frac{x+1}{x-2} - 4\right| = 3|x-3| \cdot \frac{1}{|x-2|} < 3|x-3| \cdot 2 = 6|x-3|.$$

   Let $\delta < \min(1/2, \varepsilon/6)$. Then when $|x-3| < \delta$ we have:

$$\left|\frac{x+1}{x-2} - 4\right| < 6|x-3| < 6 \cdot \frac{\varepsilon}{6} = \varepsilon$$

   as desired.

10) By simplifying a bit we can notice that this is the same as $\frac{1}{x-2}$, so the discontinuity will be at $x = 2$. Let $x_n = 2 + 1/n$, then we have $x_n \to 2$, but:

$$f(x_n) = \frac{1}{(2 + 1/n) - 2} = n$$

and this does not converge to $f(2)$, so $f(x)$ is not continuous.

11) First we'll show $\lim_{x\to 0} f(x) = 0$. Let $\varepsilon > 0$ be given, we will find $\delta > 0$ such that:

$$0 < |x| < \delta \implies |f(x)| < \varepsilon.$$

Since $f(x)$ is always either $x$ or 0, we will always have $|f(x)| \le |x|$. Thus we can take any $\delta < \varepsilon$, and when $|x| < \delta$ we have

$$|f(x)| \le |x| < \delta = \varepsilon$$

as desired.

Now we'll show $\lim_{x\to 1} f(x)$ does not exist. Let $x_n$ be a sequence of rationals converging to 1, and let $y_n$ be a sequence of irrationals converging to 1. Then $x_n$ and $y_n$ have the same limit, but $f(x_n) = x_n \to 1$, while $f(y_n) = 0 \to 0$. Since $f(x_n)$ and $f(y_n)$ have different limits, $f$ is not continuous.

12) We'll show it's continuous at 0. I'll just write out the red part:

$$\left|\frac{x+1}{x-2} + \frac{1}{2}\right| = \left|\frac{2(x+1) + (x-2)}{2(x-2)}\right| = \frac{3}{4}|x| \cdot \frac{1}{|x-2|}$$

Now if $\delta < 1$, then $|x| < 1$ means $|x-2| > 1$ so $\frac{1}{|x-2|} < 1$. Then the above becomes

$$\left|\frac{x+1}{x-2} + \frac{1}{2}\right| < \frac{3}{4}|x| \cdot 1.$$

Then we let $\delta < \min(1, \frac{4}{3}\varepsilon)$.

13) It is discontinuous at $x = 2$. Let $x_n = 2 + 1/n$ and $y_n = 2 - 1/n$. Then $x_n$ and $y_n$ have the same limit. We'll show $f(x_n)$ and $f(y_n)$ have different limits. We have $f(x_n) = f(2 + 1/n) = (2 + 1/n)^2 \to 2^2 = 4$ and $f(y_n) = f(2 - 1/n) = 3(2 - 1/n) + 5) \to 3(2) + 5 = 11$ as desired.

14) First we'll show it's discontinuous at $c \in \mathbb{Z}$. Do this using sequences $x_n = c + 1/n$ and $y_n = c - 1/n$. I'll leave the details to you.

Now we'll show it's continuous when $c \notin \mathbb{Z}$. Let $\varepsilon > 0$ be given, we will find $\delta > 0$ such that:

$$0 < |x - c| < \delta \implies |[[x]] - [[c]]| < \varepsilon.$$

We need to choose $\delta$ so small that when $x$ is within $\delta$ of $c$, we will have $[[x]] = [[c]]$. It will suffice to choose $\delta < \min(c - [[c]], [[c]] + 1 - c)$. (Look on a picture- this makes $\delta$ smaller than the distance from $c$ to the nearest integer.) Now if $|x - c| < \delta$ we'll have $[[x]] = [[c]]$ and so:

$$|[[x]] - [[c]]| = 0 < \varepsilon$$

as desired.

15) $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. It's continuous because it's a polynomial. (We proved any polynomial is continuous. Otherwise you can show directly using the $\varepsilon$s.) To show it's not uniformly continuous, let $x_n = n$ and $y_n = n + \frac{1}{n}$. Then we can check that $|x_n - y_n| \to 0$, but:

$$|f(x_n) - f(y_n)| = |n^2 - (n - \frac{1}{n})^2| = |2 - \frac{1}{n^2}| \to 2 \neq 0$$

so $f$ is not uniformly continuous on $\mathbb{R}$.

16) Let $x_n = 2 + 1/n$ and $y_n = 2 - 1/n$. Then $|x_n - y_n| \to 0$, but we have

$$|f(x_n) - f(y_n)| = \left|\frac{2+1/n+1}{2+1/n-2} - \frac{2-1/n+1}{2-1/n-2}\right| = |3n+1+(3n-1)| = |6n| \not\to 0$$

17) Let $\varepsilon > 0$ be given, we will find $\delta > 0$ such that for all $x, y \in [-5, -1]$ we have:

$$0 < |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

We simplify:

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - 6x + 3 - (y^2 - 6y + 3)| = |x^2 - y^2 - 6x + 6y| \leq |x^2 - y^2| + |-6x - 6y| \\ &= |x-y||x+y| + 6|x-y| = |x-y|(|x+y| + 6) \end{aligned}$$

and $|x+y| \leq |-5 + -5| = 10$ since $x, y \in [-5, -1]$, so we have $|f(x) - f(y)| < 16|x-y|$.
Let $\delta = \frac{\varepsilon}{16}$. Then when $|x-y| < \delta$ we have:

$$|f(x) - f(y)| = |x^2 - 6x + 3 - (y^2 - 6y + 3)| \leq 16|x-y| < 16 \cdot \frac{\varepsilon}{16} = \varepsilon$$

as desired.