Math 3371 final exam topics & practice

Old topics

- Bounded sets, Axiom of Completeness, sup & inf
- Nested intervals property, Archimedean property
- Cardinality
- Sequences & convergence
- Monotone convergence theorem, Bolzano-Weierstrauss theorem
- Open & closed sets
- Compact sets
- Connected sets
- Limits of functions
- Continuity
- Uniform continuity

New topics (at least half of the exam)

Intermediate value theorem

1) Use the IVT to show that $\sqrt{2}$ exists. (i.e. show that $x^2 - 2 = 0$ has a solution.)
2) Let $f, g : [0, 1] \to \mathbb{R}$ be two continuous functions with $f(0) < g(0)$ and $f(1) > g(1)$. Show that there is some $c \in [0, 1]$ with $f(c) = g(c)$.
3) Consider 3 functions in the above scenario with $f(0) < g(0) < h(0)$ and $f(1) > g(1) > h(1)$. Must there be some point c with $f(c) = g(c) = h(c)$? Either prove it or show a counterexample. (You can just draw a graph for an example if you like.)

The derivative

4) Show in detail that $f(x) = |x|$ is differentiable whenever $x \neq 0$ and is not differentiable when $x = 0$.
5) Show that this is differentiable at $x = 0$:

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that it is not differentiable at $x = 0$ if x^2 is changed to x. What about $x^{3/2}$?

6) For a differentiable function f, we say f is *uniformly* differentiable on a set A when: for all $\epsilon > 0$ there exists $\delta > 0$ such that, for every $x, y \in A$ we have:

$$0 < |x - y| < \delta \implies \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon.$$

Show that $f(x) = x^2$ is uniformly differentiable on $\mathbb{R}$. What about x^3? (prove it either way)

Mean value theorem

7) Use MVT to prove Rolle's theorem: if $f : [a, b] \to \mathbb{R}$ is differentiable and $f(a) = f(b)$, then there is some $c \in (a, b)$ with $f'(c) = 0$.

8) If f, g are as in problem b) from IVT above, must there be some point where $f'(c) = g'(c)$? Prove it or give a counterexample.

9) Say f is continuous and f' is continuous, and we know these values:

$$f(0) = 1, f(3) = 5, f(4) = 1$$

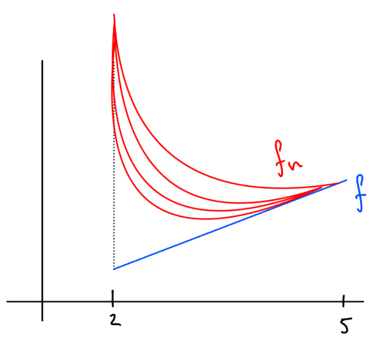
For which $d \in \mathbb{R}$ will we be guaranteed to find $c \in (0, 4)$ with $f'(c) = d$? Is it possible that $f'(c) = 10$ for some $c \in (0, 4)$?

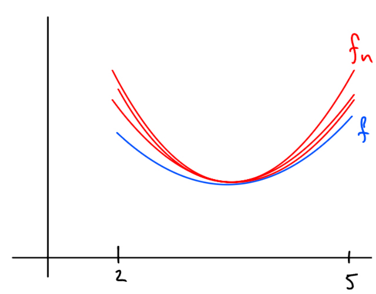
Sequences of functions

10) Let $f_n(x) = x^2 + \frac{x}{n}$. Find f such that $f_n \to f$ pointwise on $\mathbb{R}$.

11) Let $g_n(x) = \frac{1}{x - 1/n}$. Find g such that $g_n \to g$, and prove that it converges uniformly on $(2, 5)$.

12) Be able to tell based on a picture of functions whether or not f_n converges uniformly. For example, on each of these pictures, is the convergence uniform on $(2, 5)$?



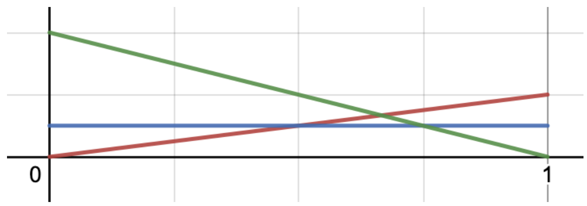


Answers!

1) Let $f(x) = x^2 - 2$, which is continuous, so we can use MVT. We can check that $f(0) = -2$ and $f(2) = 2$, so we have $f(0) < 0 < f(2)$. Thus by MVT there is some $c \in (0, 2)$ with $f(c) = 0$ as desired.

2) Let $h(x) = f(x) - g(x)$, so $h : [0, 1] \to \mathbb{R}$ is continuous. Then $h(0) = f(0) - g(0) < 0$, and $h(1) = f(1) - g(1) > 0$. So $h(0) < 0 < h(1)$, so there is some point $c \in (0, 1)$ with $h(c) = 0$, which means $f(c) = g(c)$.

3) No- there need not be such a point. For example consider 3 functions that look like this:



The lines cross over one another, but there is no single point where all 3 meet.

4) When $x \neq 0$: Let $c \neq 0$, we'll show $f(x)$ is differentiable at c. We have:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{|x| - |c|}{x - c}$$

We'll prove that this is either 1 or -1 using the epsilons.

Assuming that $c > 0$, we'll show the limit above is $+1$. The same argument will show it's -1 when $c < 0$. Let $\epsilon > 0$ be given, we'll find $\delta > 0$ such that

$$0 < |x - c| < \delta \implies \left| \frac{|x| - |c|}{x - c} - 1 \right| < \epsilon.$$

Choose δ so small that $|x - c| < \delta$ implies x and c have the same sign. (We can use $\delta = |c|$ to make this happen.) Then since x and c have the same sign and c is positive, they are both positive and so we'll have $|x| - |c| = x - c$. Then we get:

$$\left| \frac{|x| - |c|}{x - c} - 1 \right| = \left| \frac{x - c}{x - c} - 1 \right| = 0 < \epsilon$$

as desired.

Now when $x = 0$: We must show $f(x)$ is not differentiable at 0, that is, we'll show

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist. That is, we must show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist. We do this with sequences: consider $x_n = 1/n$ and $y_n = -1/n$. Then x_n and y_n both converge to 0, but

$$\frac{|x_n|}{x_n} = \frac{1/n}{1/n} = 1$$

and

$$\frac{|y_n|}{y_n} = \frac{1/n}{-1/n} = -1$$

so x_n and y_n have the same limit, but when plugged in they give different limits. Thus $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

5) We will show that $f'(0) = 0$. That is, we must show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

Plugging in the function and simplifying, this means we must show:

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0$$

We do this with the epsilons. Let $\epsilon > 0$ be given, we'll find $\delta > 0$ such that:

$$0 < |x| < \delta \implies |x \sin(1/x)| < \epsilon.$$

$|x \sin(1/x)| = |x||\sin(1/x)| \leq |x|$

Let $\delta = \epsilon$. Then if $0 < |x| < \delta$, we have:

$$|x \sin(1/x)| \leq |x| < \epsilon$$

as desired.

What about when x^2 is changed to x? Same stuff as above, but now the limit becomes just

$$\lim_{x \to 0} \sin(1/x)$$

instead of $x \sin(1/x)$. This limit does not exist- we can show this with sequences. Let $x_n = \frac{1}{2\pi n}$, and $y_n = \frac{1}{2\pi n + \pi/2}$. Then both these sequences converge to 0, but when we plug them in we get:

$$\sin(1/x_n) = \sin(2\pi n) = 0$$

and

$$\sin(1/y_n) = \sin(2\pi n + \pi/2) = 1$$

so the limit above does not exist.

For $x^{3/2}$, it is differentiable at $x = 0$. The appropriate limit becomes $\lim_{x \to 0} x^{1/2} \sin(1/x) = 0$, and we prove this with the epsilons just like we did when it was x^2. (You should say: let $\delta = \epsilon^2$.)

6) Let $\epsilon > 0$ be given, we will find $\delta > 0$ such that, for all $x, y \in \mathbb{R}$, we have:

$$0 < |x - y| < \delta \implies \left| \frac{x^2 - y^2}{x - y} - 2x \right| < \epsilon.$$

We have:

$$\left| \frac{x^2 - y^2}{x - y} - 2x \right| = \left| \frac{(x - y)(x + y)}{x - y} - 2x \right| = |x + y - 2x| = |y - x|$$

Let $\delta = \epsilon$. Then if $|x - y| < \delta$, we have:

$$\left| \frac{x^2 - y^2}{x - y} - 2x \right| = |y - x| = |x - y| < \epsilon$$

as desired.

For $f(x) = x^3$, it is not uniformly differentiable on $\mathbb{R}$. We do this with two sequences similarly to showing that something is not uniformly continuous. Let $x_n = n$ and $y_n = n + \frac{1}{n}$, so that $|x_n - y_n| \to 0$. Then we must show that:

$$\left| \frac{f(x_n) - f(y_n)}{x_n - y_n} - f'(x_n) \right| \not\to 0.$$

After lots and lots of simplifying (I hope I did it right), we get:

$$\left|\frac{f(x_n) - f(y_n)}{x_n - y_n} - f'(x_n)\right| = |3 + n^{-2}| \to 3 \neq 0$$

as desired.

7) Applying MVT to $f(x)$ on the interval $[a, b]$ shows that there is some $c \in (a, b)$ with:

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

as desired.

8) No! For example f and g can be two straight lines of opposite slope. There is no need for them to have matching derivatives at any point.

9) It is helpful to plot these points and look at them. The greatest slope among these points occurs between $f(0) = 1$ and $f(3) = 5$. Thus MVT will give a point with a slope of $\frac{5-1}{3-0} = 4/3$. The smallest slope among these points occurs between $f(3) = 5$ and $f(4) = 1$. Then MVT will give a point with a slope of $\frac{1-5}{4-3} = -4$. By the intermediate value theorem (applied to f') we can also achieve any slope in between these two extremes. So the possible slopes we are guaranteed to find for $c \in (0, 4)$ will be:

$$-4 \leq f'(c) \leq 4/3.$$

It is certainly possible that $f'(c) = 10$ (if there is a random spike in between the given points), but there is no guarantee that there is such a point.

10) The function f is $f(x) = x^2$. First we show $f_n \to f$ pointwise on $\mathbb{R}$. Take some specific $x \in \mathbb{R}$, and let $\epsilon > 0$ be given. We'll find $N \in \mathbb{N}$ such that

$$n > N \implies |f_n(x) - f(x)| < \epsilon.$$

We have:

$$|f_n(x) - f(x)| = |x^2 + x/n - x^2| = |x/n| = |x|/n.$$

We want this to be less than ϵ, so we need $n > \epsilon/|x|$. (If $|x| = 0$, then the stuff above equals zero, so there is nothing to prove in that case.)

Let $N > \epsilon/|x|$. Then if $n > N$, we have:

$$|f_n(x) - f(x)| = |x|/n < |x|/(\epsilon/|x|) = \epsilon$$

as desired.

The convergence is not uniform. Given any n, the distance from $x^2 + x/n$ to x^2 is $|x|/n$, and this is not bounded. So for any ϵ, it is impossible for $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$.

11) $g_n \to g$ where $g(x) = 1/x$. We'll show $g_n \to g$ uniformly on $(2, 5)$. Let $\epsilon > 0$ be given, and we will find $N \in \mathbb{N}$ such that, for all $x \in (2, 5)$, we have:

$$n > N \implies |g_n(x) - g(x)| < \epsilon.$$

We have

$$|g_n(x) - g(x)| = \left|\frac{1}{x - 1/n} - \frac{1}{x}\right| = \left|\frac{x}{x(x - 1/n)} - \frac{x - 1/n}{x(x - 1/n}\right| = \left|\frac{-1/n}{x(x - 1/n)}\right| \leq \frac{1/n}{2(2 - 1/n)}$$

where the last step is because $x \in (2, 5)$. Continuing simplifying gives:

$$|g_n(x) - g(x)| \leq \frac{1}{4n-2}$$

Let $N > \frac{1}{4}(\frac{1}{\epsilon} + 2)$. Then if $n > N$ we have:

$$|g_n(x) - g(x)| \leq \frac{1}{4n-2} < \epsilon$$

as desired.