Weighted Voting

$[10 : \underbrace{5, 4, 1, 1}_{\text{weights}}]$

quota

How can we measure each voter's power?

Here the 5 & 4 have equal power,

the 1's still matter, e.g. $5+4+1=10$

Dictator, Dummy, & Veto power

Dictator: one voter who can meet the quota by themself.

$[10 : 11, 5, 1]$

dictator

Dummy: a voter whose vote can never matter

$[10 : 5, 5, 1]$

dummy

Veto Power: a voter whose vote is necesarry to meet the quota.

$[10 : \underbrace{5, 4}_{\uparrow}, 1, 1]$

5 & 4 have veto power, 1s do not.

$[30 : 10, 10, 10, 9]$

No dictator, 9 is a dummy

each 10 has veto power.

$[30 : 10, 10, 10, 10, 9]$

No dictator, 9 is a dummy.

No veto power – no single 10 is needed to get to 30

(you can use the other 10s)

$[10 : 14, 4, 1]$ ← Dict: 14, veto: 14, dummies: 4 & 1.

$[10 : 5, 4, 3, 2]$ ← Dict: NO, veto: 5, dum: no

$[10 : 7, 3, 3, 2]$ ← Dict: no, veto: 7, dum: 2

$[10 : 7, 3, 2]$ ← Dict: no, veto: 7 & 3, dum: 2

$[10 : 14, 9, 1]$ ← quota is too small!

$$q > \frac{1}{2}(w_1 + \dots + w_N)$$

US electoral college:

$[270 : 54, 40, 30, \ldots, 3]$

No dictator

No veto

No dummies

We need a more sophisticated measure of power

a "power index"

$[10 : 5, 4, 1, 1]$

true power index is

5: 50%
4: 50%
1: 0%
1: 0%

2 ways to do it, each is a different definition of power

The <u>Shapley - Shubik</u> <u>Power Index</u>

A way to measure the true power of each voter.

The measure of power is: who can cast "the deciding vote"

In $\begin{array}{c} \quad A \; B \; C \; D \; E \\ [6 : 3, 2, 1, 1, 1] \end{array}$

Imagine they vote in order, like BACDE.

all voting yes.

Whose vote actually makes it to the quota?

BACDE: 2, 3, 1, 1, 1

C is the pivotal voter

in BACDE.

BACDE is one of many possible voting permutations

Another perm:

CEBAD: 1 1 2 3 1

B is pivotal.

Def The Shapley-Shubik (SS) power index of a voter is the fraction of the time that this voter is pivotal among all permutations.

Ex $\begin{array}{c} \quad A \quad B \quad C \\ [16 : 12, 10, 5] \end{array}$

All perms of ABC

perms	weights	pivotal
A B C	12 10 5	B
A C B	12 5 10	C
B A C	10 12 5	A
B C A	10 5 12	A
C A B	5 12 10	A
C B A	5 10 12	A

A : 4/6 = 66.6.. %

B : 1/6 = 16.6... %

C : 1/6 = 16.6... %

each one is $\dfrac{\text{\# of times this one is pivotal}}{\text{total \# of perms}}$

$$\begin{array}{c} \quad\ \ \text{A}\ \ \text{B}\ \ \text{C} \\ [12 : 8, 7, 1] \end{array}$$

perms	weights	pivotal
A B C	8 7 1	B
A C B	8 1 7	B
B A C	7 8 1	A
B C A	7 1 8	A
C A B	1 8 7	B
C B A	1 7 8	A

A: 3/6

B: 3/6

C: 0/6

[12 : 7, 6, 5]

perms	weights	pivotal
A B C	7 6 5	B
A C B	7 5 6	C
B A C	6 7 5	A
B C A	6 5 7	A
C A B	5 7 6	A
C B A	5 6 7	A

A: 4/6

B: 1/6

C: 1/6

For more than 3, there's many more perms!

If we had A B C D : How many perms?

___ ___ ___ ___

each blank gets one letter

first blank has 4 options,
next has 3,
next has 2,
last has 1 option.

Total # of options is $\underbrace{4 \times 3 \times 2 \times 1}$ = 24

called "4 factorial"

written $4! = 24$

$3! = 3 \cdot 2 \cdot 1 = 6$

Generally, for N different things,

there are $N!$ different permutations.

this is very big
when N is large

$10!$ is 3.6 million

A strange & beautiful formula ~ 1700

Stirling's formula For large N:

$$N! \approx \sqrt{2\pi N}\ \frac{N^N}{e^N}$$