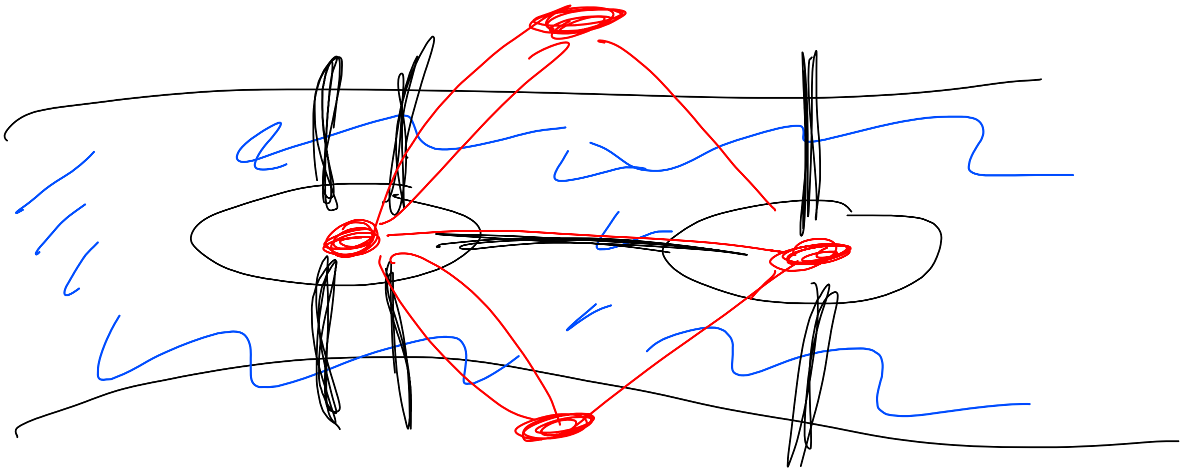
Euler circuits

OG graph theory problem

"The 7 bridges of Königsberg"

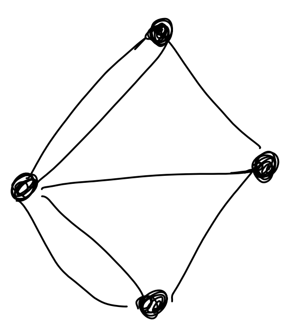


Can you walk across each bridge 1 time, no repeats, and end up where you started? NO

This is not a geometry problem

it's a graph theory problem.

About:

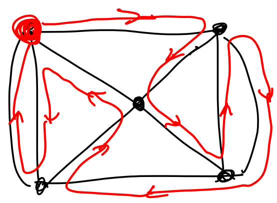


Is there a circuit which uses every edge one time?

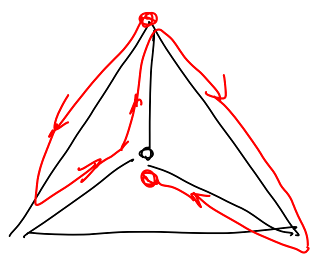
(no repeats)

Def A circuit which uses each edge exactly once is an Euler circuit.

How to tell / find if there is an Euler circuit?



This has an Euler circuit



This has no E. circuit

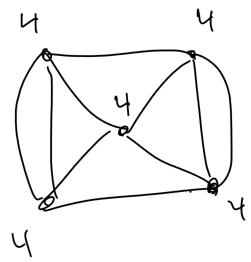
Hard to say exactly why not.

Turns out it's actually very easy to tell if a E. circuit exists or not.

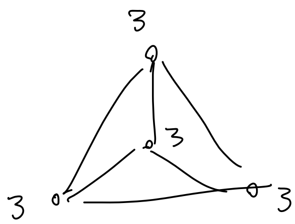
Thm A graph has an Euler circuit

if and only if — logically the same.

All verts have even degree.



All even, so there is an E. circ.

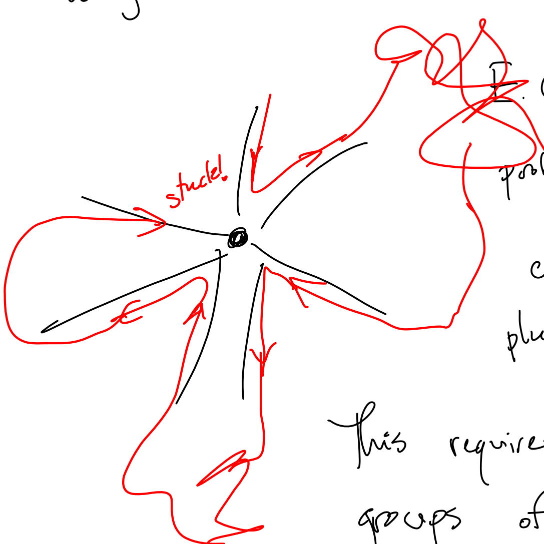


not all even, so there is no E. circuit.

The theorem tells if an E. circuit exists or not — doesn't say where it is.

"An existence theorem"

Why does Euler circ make all degs even?



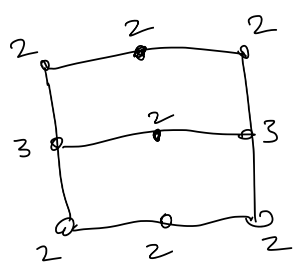
E. circ must visit each vertex, probably several times,

circuit needs an edge to arrive on, plus an edge to depart on.

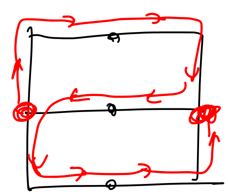
This requires all edges to come in groups of 2, i.e. degree must be even.

What about a path (not a circuit) that uses every edge exactly once?

This is an <u>Euler path</u>



Not all even, so no E. circuit.



← an E. path.

<u>Thm</u> A graph has an E. path (not a circuit) if ~~and~~ only if

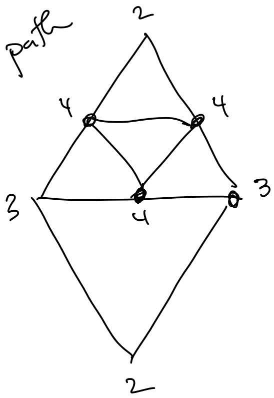
All verts have even degree <u>except</u> 2 odds.

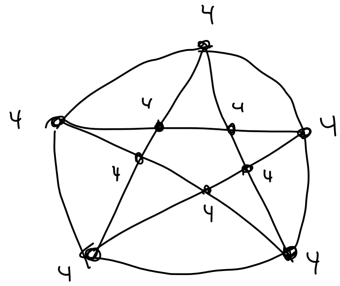
(The 2 odds must be the start & end pts of the path)

E. circ exists if all even

E. path exists if 2 odds, the rest even

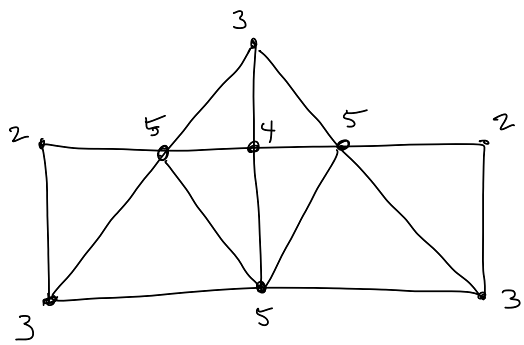
Has an Euler path



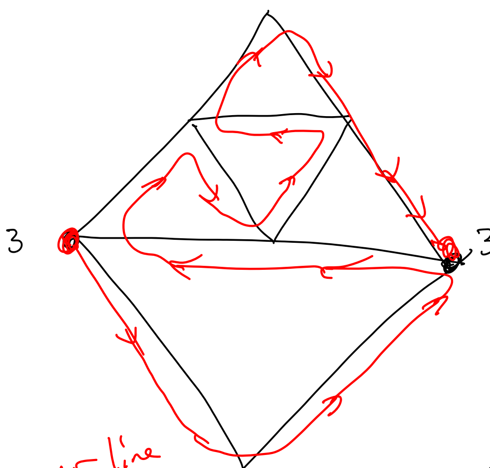


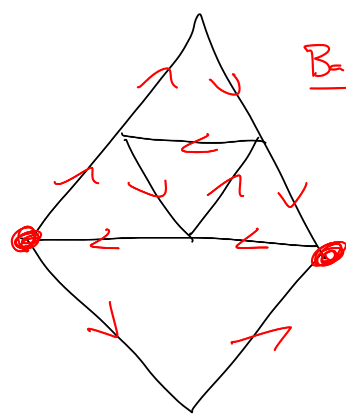
Is there a
E. circ?
E. path?

Draw it if so.

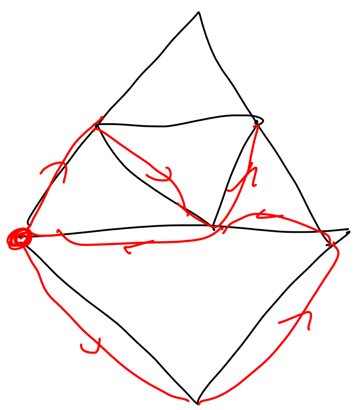


No E. circ or path.

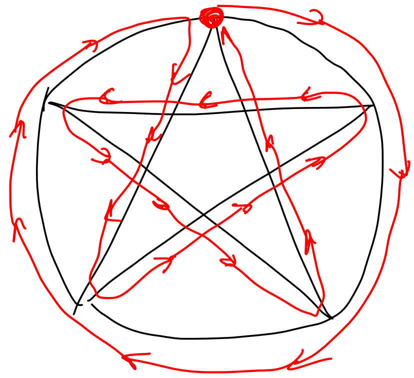




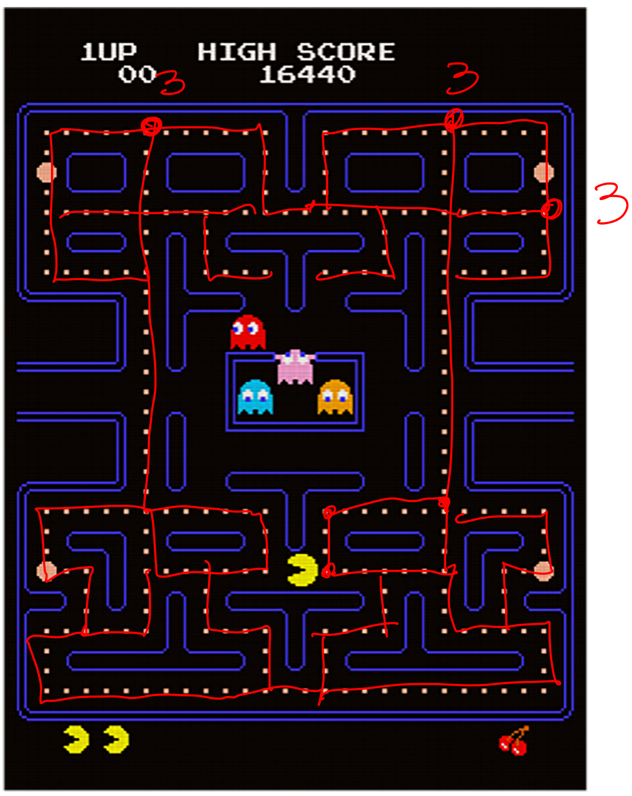
Bad



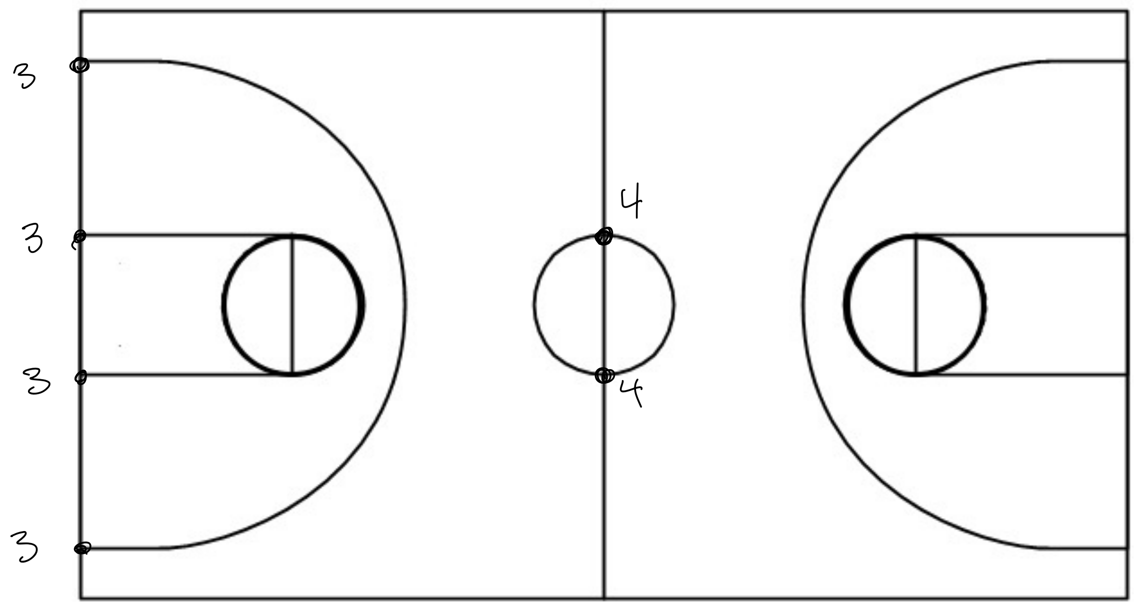
Make your line next to the edges, don't go straight thru verts.



E. circ



Is there a Euler path? NO



Many odd degrees, so no E. path or circuit.

So you can't draw all the lines in one big loop.

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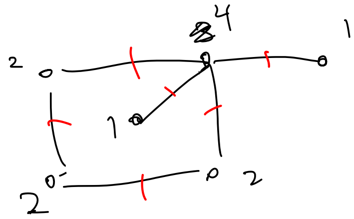
if # of odds is 0 → E. circuit

# of odds is 2 → E. path

— — — — — more than 2 → none

What if one vert is odd, the rest even?



This cannot happen.

sum = 2+4+1+1+2+2 = 12

edges = 6

Thm (Euler's sum-of-degrees theorem)

In any graph the sum of degrees equals $2 \times$ (# of edges)

This means the sum-of-degrees is always even

So we can't have 1 odd, the rest even.

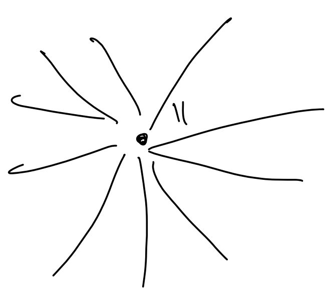
Weird NFL story:

They were deciding the schedule for the year:

there were 13 teams in the AFC

we'll play 11 games each in AFC.

in a graph:



each team is a vert, 13 verts, each of degree 11.

Total sum of degrees is $13 \times 11 = 143$

Impossible!