Math 3385: Homework #5 answers

#2

Proof. First we assume f is continuous, and show that $f^{-1}(C)$ is closed whenever $C \subset Y$ is closed. Since C is closed, $Y - C$ is open, and since f is continuous, we have $f^{-1}(Y - C)$ is open. But $f^{-1}(Y - C) = X - f^{-1}(C)$ (this is a basic fact about the complement of any preimage). So $X - f^{-1}(C)$ is open, so $f^{-1}(C)$ is closed as desired.

Now for the other direction, assume $f^{-1}(C)$ is closed whenever $C \subset Y$ is closed, and we'll show f is continuous. Let $U \subset Y$ be open, and we must show $f^{-1}(U)$ is open. Since U is open, $Y - U$ is closed, and so by our assumption about closed sets we have $f^{-1}(Y - U)$ closed. But this set equals $X - f^{-1}(U)$, so $X - f^{-1}(U)$ is closed so $f^{-1}(U)$ is open as desired. $\square$

#4a

It is continuous.

Take some basis open set $(a, b) \subset \mathbb{R}$, and we must show that $f^{-1}(a, b)$ is open in $\mathbb{R}_l$. Since f has the graph of a line, the pullback of an open interval is another open interval. So $f^{-1}(a, b) = (c, d)$ for some c and d. (It's not hard to find equations for c and d, but not necessary.)

And an open interval (c, d) is open in $\mathbb{R}_l$, so f is continuous.

#7

Proof. First assume $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous, and we'll show $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. That is, we'll show $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Take any set $U \in \mathcal{T}_2$, and we'll show $U \in \mathcal{T}_1$. Since $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous, U being open in $\mathcal{T}_2$ means that $\mathrm{id}^{-1}(U) = U$ is open in $\mathcal{T}_1$, so $U \in \mathcal{T}_1$ as desired.

Now assume that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, and we'll show that $\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous. Take U open in $\mathcal{T}_2$, and we must show that $\mathrm{id}^{-1}(U)$ is open in $\mathcal{T}_1$. But $\mathrm{id}^{-1}(U) = U$, and since $U \in \mathcal{T}_2$ and $\mathcal{T}_2 \subseteq \mathcal{T}_1$, we have $U \in \mathcal{T}_1$ as desired. $\square$

#11

Proof. We must show $f|_A : A \to Y$ is continuous, so take an open set $U \subset Y$, and we'll show $f|_A^{-1}(U)$ is open in A. We have $f|_A^{-1}(U) = f^{-1}(U) \cap A$, and since f is continuous, $f^{-1}(U)$ is open in X. Thus $f|_A^{-1}(U) = f^{-1}(U) \cap A$ is open in A (using the subspace topology) as desired. $\square$